Topological Methods in Hydrodynamics

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With 78 Illustrations

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Preface

> *. . . ad alcuno, dico, di quelli, che troppo laconicamente vorrebbero vedere, nei più angusti spazii che possibil fusse, ristretti i filosofici insegnamenti, sí che sempre si usasse quella rigida e concisa maniera, spogliata di qualsivoglia vaghezza ed ornamento, che é propria dei puri geometri, li quali né pure una parola proferiscono che dalla assoluta necessitá non sia loro suggerita.*
>
> *Ma io, all'incontro, non ascrivo a difetto in un trattato, ancorché indirizzato ad un solo scopo, interserire altre varie notizie, purché non siano totalmente separate e senza veruna coerenza annesse al principale instituto.**
>
> Galileo Galilei
> "Lettera al Principe Leopoldo di Toscana" (1623)

Hydrodynamics is one of those fundamental areas in mathematics where progress at any moment may be regarded as a standard to measure the real success of mathematical science. Many important achievements in this field are based on profound theories rather than on experiments. In turn, those hydrodynamical theories stimulated developments in the domains of pure mathematics, such as complex analysis, topology, stability theory, bifurcation theory, and completely integrable dynamical systems. In spite of all this acknowledged success, hydrodynamics with its spectacular empirical laws remains a challenge for mathematicians. For instance, the phenomenon of turbulence has not yet acquired a rigorous mathematical theory. Furthermore, the existence problems for the smooth solutions of hydrodynamic equations of a three-dimensional fluid are still open.

The simplest but already very substantial mathematical model for fluid dynamics is the hydrodynamics of an ideal (i.e., of an incompressible and inviscid) homogeneous fluid. From the mathematical point of view, a theory of such a fluid

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* " *. . . Some prefer to see the scientific teachings condensed too laconically into the smallest possible volume, so as always to use a rigid and concise manner that whatsoever lacks beauty and embellishment, and that is so common among pure geometers who do not pronounce a single word which is not of absolute necessity.*
I, on the contrary, do not consider it a defect to insert in a treatise, albeit devoted to a single aim, other various remarks, as long as they are not out of place and without coherency with the main purpose," see [Gal].

filling a certain domain is nothing but a study of geodesics on the group of diffeomorphisms of the domain that preserve volume elements. The geodesics on this (infinite-dimensional) group are considered with respect to the right-invariant Riemannian metric given by the kinetic energy.

In 1765, L. Euler [Eul] published the equations of motion of a rigid body. Eulerian motions are described as geodesics in the group of rotations of three-dimensional Euclidean space, where the group is provided with a left-invariant metric. In essence, the Euler theory of a rigid body is fully described by this invariance. The Euler equations can be extended in the same way to an arbitrary group. As a result, one obtains, for instance, the equations of a rigid body motion in a high-dimensional space and, especially interesting, the Euler equations of the hydrodynamics of an ideal fluid.

Euler's theorems on the stability of rotations about the longest and shortest axes of the inertia ellipsoid have counterparts for an arbitrary group as well. In the case of hydrodynamics, these counterparts deliver nonlinear generalizations of Rayleigh's theorem on the stability of two-dimensional flows without inflection points of the velocity profile.

The description of ideal fluid flows by means of geodesics of the right-invariant metric allows one to apply the methods of Riemannian geometry to the study of flows. It does not immediately imply that one has to start by constructing a consistent theory of infinite-dimensional Riemannian manifolds. The latter encounters serious analytical difficulties, related in particular to the absence of existence theorems for smooth solutions of the corresponding differential equations.

On the other hand, the strategy of applying geometric methods to the infinite-dimensional problems is as follows. Having established certain facts in the finite-dimensional situation (of geodesics for invariant metrics on finite-dimensional Lie groups), one uses the results to *formulate* the corresponding facts for the infinite-dimensional case of the diffeomorphism groups. These final results often can be proved directly, leaving aside the difficult questions of foundations for the intermediate steps (such as the existence of solutions on a given time interval). The results obtained in this way have an *a priori* character: the derived identities or inequalities take place for any reasonable meaning of "solutions," provided that such solutions exist. The actual existence of the solutions remains an open question.

For example, we deduce the formulas for the Riemannian curvature of a group endowed with an invariant Riemannian metric. Applying these formulas to the case of the infinite-dimensional manifold whose geodesics are motions of the ideal fluid, we find that the curvature is negative in many directions. Negativeness of the curvature implies instability of motion along the geodesics (which is well-known in Riemannian geometry of finite-dimensional manifolds). In the context of the (infinite-dimensional) case of the diffeomorphism group, we conclude that the ideal flow is unstable (in the sense that a small variation of the initial data implies large changes of the particle positions at a later time). Moreover, the curvature formulas allow one to estimate the increment of the exponential deviation of fluid particles with close initial positions and hence to pre-

dict the time period when the motion of fluid masses becomes essentially unpredictable.

For instance, in the simplest and utmost idealized model of the earth's atmosphere (regarded as two-dimensional ideal fluid on a torus surface), the deviations grow by the factor of 10^5 in 2 months. This circumstance ensures that a dynamical weather forecast for such a period is practically impossible (however powerful the computers and however dense the grid of data used for this purpose).

The table of contents is essentially self explanatory. We have tried to make the chapters as independent of each other as possible. Cross-references within the same chapter do not contain the chapter number.

For a first acquaintance with the subject, we address the reader to the following sections in each chapter: Sections I.1–5 and I.12, Sections II.1 and II.3–4, Sections III.1–2 and III.4, Section IV.1, Sections V.1–2, Sections VI.1 and VI.4.

Some statements in this book may be new even for the experts. We mention the classification of the local conservation laws in ideal hydrodynamics (Theorem I.9.9), M. Freedman's solution of the A. Sakharov–Ya. Zeldovich problem on the energy minimization of the unknotted magnetic field (Theorem III.3), a discussion of the construction of manifold invariants from the energy bounds (Remark III.2.6), a discussion of a complex version of the Vassiliev knot invariants (in Section III.7.E), a nice remark of B. Zeldovich on the Lobachevsky triangle medians (Problem IV.1.4), the relation of the covariant derivative of a vector field and the inertia operator in hydrodynamics (Section IV.1.D), a digression on the Fokker–Planck equation (Section V.3.C), and the dynamo construction from the geodesic flow on surfaces of constant negative curvature (Section V.4.D).

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Acknowledgments

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Contents

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Chapter I

Group and Hamiltonian Structures of Fluid Dynamics

The group we will most often be dealing with in hydrodynamics is the infinite-dimensional group of diffeomorphisms that preserve the volume element of the domain of a fluid flow. One can also relate many rather interesting systems to other groups, in particular, to finite-dimensional ones. For example, the ordinary theory of a rigid body with a fixed point corresponds to the rotation group $SO(3)$, while the Lobachevsky geometry has to do with the group of translations and dilations of a vector space. Our constructions are equally applicable to the gauge groups exploited by physicists. The latter groups occupy an intermediate position between the rotation group of a rigid body and the diffeomorphism groups. They are already infinite-dimensional but yet too simple to serve as a model for hydrodynamics.

In this chapter we study geodesics of one-sided invariant Riemannian metrics on Lie groups. The principle of least action asserts that motions of physical systems such as rigid bodies and ideal fluids are described by the geodesics in these metrics given by the kinetic energy.

§1. Symmetry groups for a rigid body and an ideal fluid

Definition 1.1. A set G of smooth transformations of a manifold M into itself is called a *group* if

> (i) along with every two transformations $g, h \in G$, the composition $g \circ h$ belongs to G (the symbol $g \circ h$ means that one first applies h and then g);
>
> (ii) along with every $g \in G$, the inverse transformation g^{-1} belongs to G as well.

From (i) and (ii) it follows that every group contains the identity transformation (the unity) e.

A group is called a *Lie group* if G has a smooth structure and the operations (i) and (ii) are smooth.

Example 1.2. All rotations of a rigid body about the origin form the Lie group $SO(3)$.

Example 1.3. Diffeomorphisms preserving the volume element in a domain M form a Lie group. Throughout the book we denote this group by $S\,\mathrm{Diff}(M)$ (or by $\mathcal{D}$ to avoid complicated formulas).

The group $S\,\mathrm{Diff}(M)$ can be regarded as the configuration space of an incompressible fluid filling the domain M. Indeed, a fluid flow determines for every time moment t the map g^t of the flow domain to itself (the initial position of every fluid particle is taken to its terminal position at the moment t). All the terminal positions, i.e., configurations of the system (or "permutations of particles"), form the "infinite-dimensional manifold" $S\,\mathrm{Diff}(M)$. (Here and in the sequel we consider only the diffeomorphisms of M that can be connected with the identity transformation by a continuous family of diffeomorphisms. Our notation $S\,\mathrm{Diff}(M)$ stands only for the connected component of the identity of the group of all volume-preserving diffeomorphisms of M.)

The kinetic energy of a fluid (under the assumption that the fluid density is 1) is the integral (over the flow domain) of half the square of the velocity of the fluid particles. Since the fluid is incompressible, the integration can be carried out either with the volume element occupied by an initial particle or with the volume element dx occupied by that at the moment t:

$$E = \frac{1}{2} \int_M v^2 \, dx,$$

where v is the velocity field of the fluid: $v(x, t) = \frac{\partial}{\partial t} g^t(y)$, $x = g^t(y)$ (y is an initial position of the particle whose position is x at the moment t), see Fig. 1.
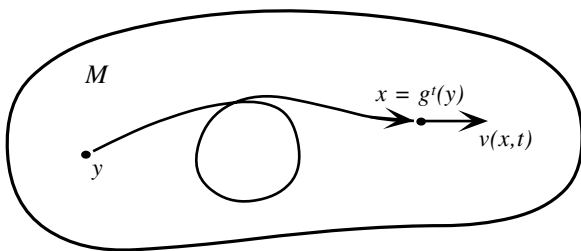
FIGURE 1.   The motion of a fluid particle in a domain M.

Suppose that a configuration g changes with velocity $\dot{g}$. The vector $\dot{g}$ belongs to the tangent space $T_g G$ of the group $G = S\,\mathrm{Diff}(M)$ at the point g. The kinetic energy is a quadratic form on this vector space of velocities.

Theorem 1.4. *The kinetic energy of an incompressible fluid is invariant with respect to the* right *translations on the group $G = S\,\mathrm{Diff}(M)$ (i.e., with respect to the mappings $R_h : G \to G$ of the type $R_h(g) = gh$).*

Proof. The multiplication of all group elements by h from the *right* means that the diffeomorphism h (preserving the volume element) acts *first*, before a diffeomorphism g changing with the velocity $\dot{g}$. Such a diffeomorphism h can be regarded as a (volume-preserving) renumeration of particles at the initial position, $y = h(z)$. The velocity of the particle occupying a certain position at a given moment does not change under the renumeration, and therefore the kinetic energy is preserved.   □

Similarly, the kinetic energy of a rigid body fixed at some point is a quadratic form on every tangent space to the configuration space of the rigid body, i.e., to the manifold $G = SO(3)$.

Theorem 1.5. *The kinetic energy of a rigid body is invariant with respect to the* left *translations on the group $G = SO(3)$, i.e., with respect to the transformations $L_h : G \to G$ having the form $L_h(g) = hg$.*

Proof. The multiplication of the group elements by h from the *left* means that the rotation h is carried out *after* the rotation g, changing with the velocity $\dot{g}$. Such a rotation h can be regarded as a revolution of the entire space, along with the rotating body. This revolution does not change the length of the velocity vector of each point of the body, and hence it does not change the total kinetic energy.   □

Remark 1.6. On the group $SO(3)$ (and more generally, on every compact group) there exists a two-sided invariant metric. On the infinite-dimensional groups of most interest for hydrodynamics, there is no such Riemannian metric. However, for two- and three-dimensional hydrodynamics, on the corresponding groups of volume-preserving diffeomorphisms there are two-sided invariant nondegenerate quadratic forms in every tangent space (see Section IV.8.C for the two-dimensional case, and Sections III.4 and IV.8.D for three dimensions, where this quadratic form is "helicity").

§2. Lie groups, Lie algebras, and adjoint representation

In this section we set forth basic facts about Lie groups and Lie algebras in the form and with the notations used in the sequel.

A linear coordinate change C sends an operator matrix B to the matrix CBC^{-1}. A similar construction exists for an arbitrary Lie group G.

Definition 2.1. The composition $A_g = R_{g^{-1}}L_g : G \to G$ of the right and left translations, which sends any group element $h \in G$ to ghg^{-1}, is called an *inner automorphism* of the group G. (The product of $R_{g^{-1}}$ and L_g can be taken in either order: all the left translations commute with all the right ones.) It is indeed an automorphism, since

$$A_g(fh) = (A_g f)(A_g h).$$

The map sending a group element g to the inner automorphism A_g is a group homomorphism, since $A_{gh} = A_g A_h$.

The inner automorphism A_g does not affect the group unity. Hence, its derivative at the unity takes the tangent space to the group at the unity to itself.

Definition 2.2. The tangent space to the Lie group at the unity is called the *vector space of the Lie algebra* corresponding to the group.

The Lie algebra of a group G is usually denoted by the corresponding Gothic letter $\mathfrak{g}$.

Example 2.3. For the Lie group $G = S\,\mathrm{Diff}\,(M)$, formed by the diffeomorphisms preserving the volume element of the flow domain M, the corresponding Lie algebra consists of divergence-free vector fields in M.

Example 2.4. The Lie algebra $\mathfrak{so}(n)$ of the rotation group $SO(n)$ consists of skew-symmetric $n \times n$ matrices. For $n = 3$ the vector space of skew-symmetric matrices is three-dimensional. The vectors of this three-dimensional space are said to be *angular velocities*.

Definition 2.5. The differential of the inner automorphism A_g at the group unity e is called the *group adjoint operator* Ad_g:

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \qquad \mathrm{Ad}_g\, a = (A_{g*}\big|_e)a, \qquad a \in \mathfrak{g} = T_e G.$$

(Here and in the sequel, we denote by $T_x M$ the tangent space of the manifold M at the point x, and by $F_*\big|_x : T_x M \to T_{F(x)} M$ the derivative of the mapping $F : M \to M$ at x. The derivative F_* of F at x is a linear operator.)

The adjoint operators form a representation of the group: $\mathrm{Ad}_{gh} = \mathrm{Ad}_g\, \mathrm{Ad}_h$ by the linear operators acting in the Lie algebra space.

Example 2.6. The adjoint operators of the group $S\,\mathrm{Diff}\,(M)$ define the diffeomorphism action on divergence-free vector fields in M as the coordinate changes in the manifold.

The map Ad, which associates the operator Ad_g to a group element $g \in G$, may be regarded as a map from the group to the space of the linear operators in the Lie algebra.

Definition 2.7. The differential ad of the map Ad at the group unity is called the *adjoint representation of the Lie algebra*:

$$\mathrm{ad} = \mathrm{Ad}_{*e} : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}, \qquad \mathrm{ad}_\xi = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{g(t)},$$

where $g(t)$ is a curve on the group G issued from the point $g(0) = e$ with the velocity $\dot{g}(0) = \xi$ (Fig. 2). Here, End $\mathfrak{g}$ is the space of linear operators taking $\mathfrak{g}$ to

itself. The symbol ad_ξ stands for the image of an element ξ, from the Lie algebra $\mathfrak{g}$, under the action of the linear map ad. This image $\mathrm{ad}_\xi \in \mathrm{End}\,\mathfrak{g}$ is itself a linear operator in $\mathfrak{g}$.
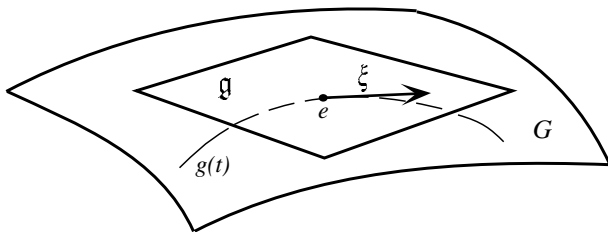
FIGURE 2.   The vector ξ in the Lie algebra $\mathfrak{g}$ is the velocity at the identity e of a path $g(t)$ on the Lie group G.

Example 2.8. Let G be the rotation group in $\mathbb{R}^n$. Then

$$\mathrm{ad}_\xi\,\omega = [\xi, \omega],$$

where $[\xi, \omega] = \xi\omega - \omega\xi$ is the commutator of skew-symmetric matrices ξ and ω. In particular, for $n = 3$ the vector $[\xi, \omega]$ is the ordinary cross product $\xi \times \omega$ of the angular velocity vectors ξ and ω in $\mathbb{R}^3$.

Proof. Let $t \mapsto g(t)$ be a curve issuing from e with the initial velocity $\dot{g} = \xi$, and let $s \mapsto h(s)$ be such a curve with the initial velocity $h' = \omega$. Then

$$g(t)h(s)g(t)^{-1} = (e + t\xi + o(t))(e + s\omega + o(s))(e + t\xi + o(t))^{-1}$$
$$= e + s[\omega + t(\xi\omega - \omega\xi) + o(t)] + o(s)$$

as $t, s \to 0$. $\square$

Example 2.9. Let $G = \mathrm{Diff}\,(M)$ be the group of diffeomorphisms of a manifold M. Then

$$(2.1) \qquad\qquad\qquad \mathrm{ad}_v\,w = -\{v, w\},$$

where $\{v, w\}$ is the Poisson bracket of vector fields v and w.

The *Poisson bracket of vector fields* is defined as the commutator of the corresponding differential operators:

$$(2.2) \qquad\qquad\qquad L_{\{v,w\}} = L_v L_w - L_w L_v.$$

The linear first-order differential operator L_v, associated to a vector field v, is the derivative along the vector field v ($L_v f = \sum v_i \frac{\partial f}{\partial x_i}$ for an arbitrary function f and any coordinate system).

The components of the field $\{v, w\}$ in an arbitrary coordinate system are expressed in terms of the components of w and v according to the following formula:

$$\{v, w\}_i = \sum_j v_j \frac{\partial w_i}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j}.$$

It follows from the above that the field $\{v, w\}$ does not depend on the coordinate system $(x_1, \ldots, x_n)$ used in the latter formula.

The operator L_v (called the *Lie derivative*) also acts on any tensor field on a manifold, and it is defined as the "fisherman derivative": the flow is transporting the tensors in front of the fisherman, who is sitting at a fixed place and differentiates in time what he sees. For instance, the functions are transported backwards by the flow, and hence $L_v f = \sum v_i \frac{\partial f}{\partial x_i}$. Similarly, differential forms are transported backwards, but vector fields are transported forwards. Thus, for vector fields we obtain that $L_v w = -\{v, w\}$.

The minus sign enters formula (2.1) since, traditionally, the sign of the Poisson bracket of two vector fields is defined according to (2.2), similar to the matrix commutator. The opposite signs in the last two examples result from the same reason as the distinction in invariance of the kinetic energy: It is left invariant in the former case and right invariant in the latter.

Proof of Formula (2.1). Diffeomorphisms corresponding to the vector fields v and w can be written (in local coordinates) in the form

$$g(t): \quad x \mapsto x + tv(x) + o(t), \ t \to 0,$$
$$h(s): \quad x \mapsto x + sw(x) + o(s), \ s \to 0.$$

Then we have $g(t)^{-1}: x \mapsto x - tv(x) + o(t)$, whence

$$h(s)(g(t))^{-1}: \quad x \mapsto x - tv(x) + o(t) + sw(x - tv(x) + o(t)) + o(s)$$
$$= \ x - tv(x) + o(t) + s\left(w(x) - t\frac{\partial w}{\partial x}v(x) + o(t)\right) + o(s),$$

and

$$g(t)h(s)(g(t))^{-1}: x \mapsto x + s\left(w(x) + t\left(\frac{\partial v}{\partial x}w(x) - \frac{\partial w}{\partial x}v(x)\right)\right) + o(t) + o(s).$$

$\square$

Example 2.10. Let $G = S\operatorname{Diff}(M)$ be the group of diffeomorphisms preserving the volume element in a domain M. Formula (2.1) is valid in this case, while all the three fields v, w, and $\{v, w\}$ are divergence free.

Definition 2.11. The *commutator* in the Lie algebra $\mathfrak{g}$ is defined as the operation $[\,,\,]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that associates to a pair of vectors a, b of the tangent space $\mathfrak{g}$ (at the unity of a Lie group G) the following third vector of this space:

$$[a, b] = \operatorname{ad}_a b.$$

The tangent space at the unity of the Lie group equipped with such operation [,] is called the *Lie algebra of the Lie group G*.

Example 2.12. The commutator of skew-symmetric matrices a and b is $ab - ba$ (in the three-dimensional case it is the cross product $a \times b$ of the corresponding vectors). The commutator of two vector fields is minus their Poisson bracket. The commutator of divergence-free vector fields in a three-dimensional Euclidean space is given by the formula

$$[a, b] = \mathrm{curl}(a \times b),$$

where $a \times b$ is the cross product. It follows from the more general formula

$$\mathrm{curl}(a \times b) = [a, b] + a \operatorname{div} b - b \operatorname{div} a,$$

and it is valid for an arbitrary Riemannian three-dimensional manifold M^3. The latter formula may be obtained by the repeated application of the homotopy formula (see Section 7.B).

Remark 2.13. The commutation operation in any Lie algebra can be defined by the following construction. Extend the vectors v and w in the left-invariant way to the entire Lie group G. In other words, at every point $g \in G$, we define a tangent vector $v_g \in T_g G$, which is the left translation by g of the vector $v \in \mathfrak{g} = T_e G$. We obtain two *left-invariant* vector fields $\tilde{v}$ and $\tilde{w}$ on G. Take their Poisson bracket $\tilde{u} = \{\tilde{v}, \tilde{w}\}$. The Poisson bracket operation is invariant under the diffeomorphisms. Hence the field $\tilde{u}$ is also *left*-invariant, and it is completely determined by its value u at the group unity. The latter vector $u \in T_e G = \mathfrak{g}$ can be taken as the definition of the commutator in the Lie algebra $\mathfrak{g}$:

$$[v, w] = u.$$

The analogous construction carried out with *right-invariant* fields $\tilde{v}$, $\tilde{w}$ on the group G provides us with *minus* the commutator.

Theorem 2.14. *The commutator operation* [,] *is bilinear, skew-symmetric, and satisfies the Jacobi identity:*

$$[\lambda a + \nu b, c] = \lambda[a, c] + \nu[b, c];$$
$$[a, b] = -[b, a];$$
$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0.$$

Remark 2.15. A vector space equipped with a bilinear skew-symmetric operation satisfying the Jacobi identity is called an *abstract Lie algebra*. Every (finite-dimensional) abstract Lie algebra is the Lie algebra of a certain Lie group G.

Unfortunately, in the infinite-dimensional case this is not so. This is a source of many difficulties in quantum field theory, in the theory of completely integrable systems, and in other areas where the language of infinite-dimensional Lie algebras

replaces that of Lie groups (see, e.g., Section VI.1 on the Virasoro algebra and KdV equation). One can view a Lie algebra as the first approximation to a Lie group, and the Jacobi identity appears as the infinitesimal consequence of associativity of the group multiplication. In a finite-dimensional situation a (connected simply connected) Lie group itself can be reconstructed from its first approximation. However, in the infinite-dimensional case such an attempt at reconstruction may lead to divergent series.

It is easy to verify the following

Theorem 2.16. *The adjoint operators* $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ *form a representation of a Lie group G by the automorphisms of its Lie algebra* $\mathfrak{g}$:

$$[\mathrm{Ad}_g \xi, \mathrm{Ad}_g \eta] = \mathrm{Ad}_g[\xi, \eta], \qquad \mathrm{Ad}_{gh} = \mathrm{Ad}_g \, \mathrm{Ad}_h \,.$$

Definition 2.17. The set of images of a Lie algebra element ξ, under the action of all the operators Ad_g, $g \in G$, is called the *adjoint* (group) *orbit* of ξ.

Examples 2.18. (A) The adjoint orbit of a matrix, regarded as an element of the Lie algebra of all complex matrices, is the set of matrices with the same Jordan normal form.

(B) The adjoint orbits of the rotation group of a three-dimensional Euclidean space are spheres centered at the origin, and the origin itself.

(C) The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of the group $SL(2, \mathbb{R})$ of real matrices with the unit determinant consists of all traceless 2×2 matrices:

$$\mathfrak{sl}(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$$

with real a, b, and c. Matrices with the same Jordan normal form have equal values of the determinant $\Delta = -(a^2 + bc)$. The adjoint orbits in $\mathfrak{sl}(2, \mathbb{R})$ are defined by this determinant "almost uniquely," though they are finer than in the complex case. The orbits are the connected components of the quadrics $a^2 + bc = \mathrm{const} \neq 0$, each half of the cone $a^2 + bc = 0$, and the origin $a = b = c = 0$; see Fig. 3a.

(D) The adjoint orbits of the group $G = \{x \mapsto ax + b \mid a > 0, b \in \mathbb{R}\}$ of affine transformations of the real line $\mathbb{R}$ are straight lines $\{\alpha = \mathrm{const} \neq 0\}$, two rays $\{\alpha = 0, \beta > 0\}$, $\{\alpha = 0, \beta < 0\}$, and the origin $\{\alpha = 0, \beta = 0\}$ in the plane $\{(\alpha, \beta)\} = \mathfrak{g}$; Fig. 3b.

(E) Let v be a divergence-free vector field on M. The adjoint orbit of v for the group $S \, \mathrm{Diff}(M)$ consists of the divergence-free vector fields obtained from v by the natural action of all diffeomorphisms preserving the volume element in the domain M. In particular, all such fields are topologically equivalent. For instance,
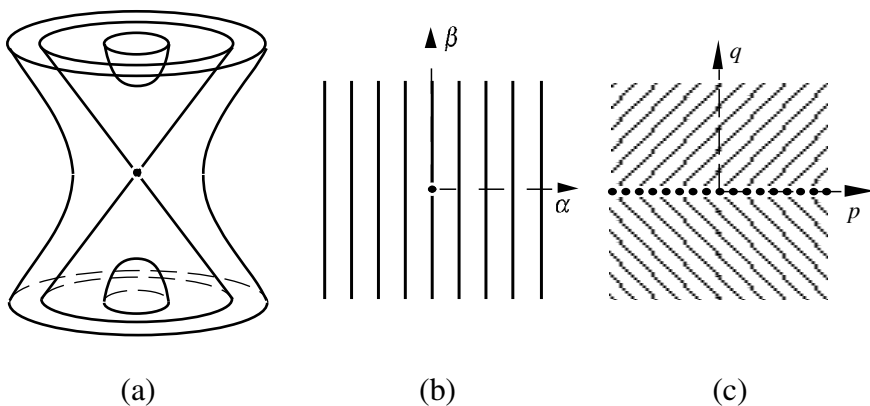
FIGURE 3.   (a) The (co)adjoint orbits in the matrix algebra $\mathfrak{sl}(2, \mathbb{R})$ are the connected components of the quadrics. The adjoint (b) and coadjoint (c) orbits of the group of affine transformations of $\mathbb{R}$.

they have equal numbers of stagnation points, of periodic orbits, of invariant surfaces, the same eigenvalues of linearizations at fixed points, etc.

Remark 2.19. For a simply connected bounded domain M in the (x, y)-plane, a divergence-free vector field tangent to the boundary of M can be defined by its *stream function* ψ (such that the field components are $-\psi_y$ and ψ_x). One can assume that the stream function vanishes on the boundary of M. The Lie algebra of the group $S\operatorname{Diff}(M)$, which consists of diffeomorphisms preserving the area element of the domain M, is naturally identified with the space of all such stream functions ψ.

Theorem 2.20. *All momenta* $I_n = \iint\limits_{M} \psi^n dxdy$ *are constant along the adjoint orbits of the group* $S\operatorname{Diff}(M)$ *in the space of stream functions.*

Proof. Along every orbit all the areas $S(c)$ of the sets "of smaller values" $\{(x, y) \mid \psi(x, y) < c\}$ are constant. $\square$

Remark 2.21. Besides the above quantities, neither a topological type of the function ψ (in particular, the number of singular points, configuration of saddle separatrices, etc.) nor the areas bounded by connected components of level curves of the stream function ψ change along the orbits; see Fig. 4.

The periods of particle motion along corresponding closed trajectories are constant under the diffeomorphism action as well. However, the latter invariant can be expressed in terms of the preceding ones. For instance, the period of motion along the closed trajectory $\psi = c$, which bounds a topological disk of area $S(c)$, is given by the formula $T = \frac{\partial S}{\partial c}$.
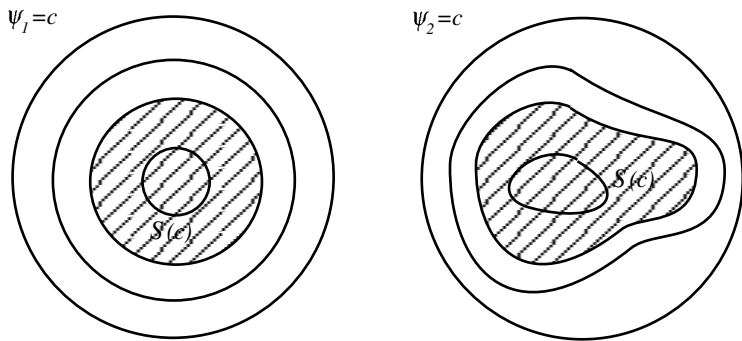
FIGURE 4.   The stream functions of the fields from the same adjoint orbit have equal areas of "smaller values" sets.

§3. Coadjoint representation of a Lie group

The main battlefield of the Eulerian hydrodynamics of an ideal fluid, as well as of the Eulerian dynamics of a rigid body, is not the Lie algebra, but the corresponding dual space, not the space of adjoint representation, but that of coadjoint representation of the corresponding group.

3.A. Definition of the coadjoint representation

Consider the vector space $\mathfrak{g}^*$ dual to a Lie algebra $\mathfrak{g}$. Vectors of $\mathfrak{g}^*$ are linear functions on the space of the Lie algebra $\mathfrak{g}$. The space $\mathfrak{g}^*$, in general, does not have a natural structure of a Lie algebra.

Example 3.1. Every *component* of the vector of angular velocity of a rigid body is a *vector* of the space dual to the Lie algebra $\mathfrak{so}(3)$.

To every linear operator $A : X \to Y$ one can associate the dual (or adjoint) operator acting in the reverse direction, between the corresponding dual spaces, $A^* : Y^* \to X^*$, and defined by the formula

$$(A^*y)(x) = y(Ax)$$

for every $x \in X$, $y \in Y^*$. In particular, the differentials of the left and right translations

$$L_{g*} : T_h G \to T_{gh} G, \qquad R_{g*} : T_h G \to T_{hg} G$$

define the dual operators

$$L_g^* : T_{gh}^* G \to T_h^* G, \qquad R_g^* : T_{hg}^* G \to T_h^* G.$$

Definition 3.2. The *coadjoint (anti)representation* of a Lie *group* G in the space $\mathfrak{g}^*$ dual to the Lie algebra $\mathfrak{g}$ is the (anti)representation that to each group element

g associates the linear transformation

$$\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$$

dual to the transformation $Ad_g : \mathfrak{g} \to \mathfrak{g}$. In other words,

$$(\mathrm{Ad}^*_g \xi)(\omega) = \xi(\mathrm{Ad}_g \omega)$$

for every $g \in G, \xi \in \mathfrak{g}^*, \omega \in \mathfrak{g}$. The operators Ad^*_g form an *anti*representation, since $\mathrm{Ad}^*_{gh} = \mathrm{Ad}^*_h \mathrm{Ad}^*_g$.

The orbit of a point $\xi \in \mathfrak{g}^*$ under the action of the coadjoint representation of a group G (in short, the *coadjoint orbit* of ξ) is the set of all points $\mathrm{Ad}^*_g \xi$ ($g \in G$) in the space $\mathfrak{g}^*$ dual to the Lie algebra $\mathfrak{g}$ of the group G.

For the group $SO(3)$ the coadjoint orbits are spheres centered at the origin of the space $\mathfrak{so}(3)^*$. They are similar to the adjoint orbits of this group, which are spheres in the space $\mathfrak{so}(3)$. However, in general, the coadjoint and adjoint representations are not alike.

Example 3.3. Consider the group G of all affine transformations of a line $G = \{x \mapsto ax + b \mid a > 0, b \in \mathbb{R}\}$. The coadjoint representation acts on the plane $\mathfrak{g}^*$ of linear functions $p\, da + q\, db$ at the group unity ($a = 1, b = 0$). The orbits of the coadjoint representation are the upper ($q > 0$) and lower ($q < 0$) half-planes, as well as every single point $(p, 0)$ of the axis $q = 0$ (see Fig. 3c).

Definition 3.4. The *coadjoint representation* of an element v of a Lie *algebra* $\mathfrak{g}$ is the rate of change of the operator $\mathrm{Ad}^*_{g_t}$ of the coadjoint group representation as the group element g_t leaves the unity $g_0 = e$ with velocity $\dot{g} = v$. The operator of the coadjoint representation of the algebra element $v \in \mathfrak{g}$ is denoted by

$$\mathrm{ad}^*_v : \mathfrak{g}^* \to \mathfrak{g}^*.$$

It is dual to the operator of the adjoint representation $\mathrm{ad}^*_v \omega(u) = \omega(\mathrm{ad}_v u) = \omega([v, u])$ for every $v \in \mathfrak{g}, u \in \mathfrak{g}, \omega \in \mathfrak{g}^*$. Given $\omega \in \mathfrak{g}^*$, the vectors $\mathrm{ad}^*_v \omega$, with various $v \in \mathfrak{g}$, constitute the tangent space to the coadjoint orbit of the point (similar to the fact that the vectors $\mathrm{ad}_v u, v \in \mathfrak{g}$ form the tangent space to the adjoint orbit of the point $u \in \mathfrak{g}$).

3.B. Dual of the space of plane divergence-free vector fields

Look at the group $G = S\,\mathrm{Diff}(M)$ of diffeomorphisms preserving the area element of a connected and simply connected bounded domain M in the $\{(x, y)\}$-plane. The corresponding Lie algebra is identified with the space of stream functions, i.e., of smooth functions in M vanishing on the boundary. The identification is natural in the sense that it does not depend on the Euclidean structure of the plane, but it relies solely on the area element μ on M.

Definition 3.5. The *inner product* of a vector v with a differential k-form ω is the $(k-1)$-form $i_v\omega$ obtained by substituting the vector v into the form ω as the first argument:

$$(i_v\omega)(\xi_1, \ldots, \xi_{k-1}) = \omega(v, \xi_1, \ldots, \xi_{k-1}).$$

Definition 3.6. The *vector field* v, with a *stream function* ψ on a surface with an area element μ, is the field obeying the condition

(3.1)
$$i_v\mu = -d\psi.$$

For instance, suppose that (x, y) are coordinates in which $\mu = dx \wedge dy$.

Lemma 3.7. *The components of the field with a stream function ψ in the above coordinate system are*

$$v_x = -\frac{\partial\psi}{\partial y}, \quad v_y = \frac{\partial\psi}{\partial x}.$$

Proof. For an arbitrary vector u, the following identity holds by virtue of the definition of $\mu = dx \wedge dy$:

$$(i_v\mu)(u) = \mu(v, u) = \begin{vmatrix} v_x & v_y \\ dx(u) & dy(u) \end{vmatrix} = (v_x\, dy - v_y\, dx)(u).$$

$\square$

Condition (3.1) determines the stream function up to an additive constant. The latter is defined by the requirement $\psi\big|_{\partial M} = 0$.

The space dual to the space of all divergence-free vector fields v can also be described by means of smooth functions on M, however, not necessarily vanishing on ∂M. Indeed, it is natural to interpret the objects dual to vector fields in M as differential 1-forms α on M. The value of the corresponding linear function on a vector field v is

$$\alpha \mid v := \iint\limits_M \alpha(v)\mu.$$

One readily verifies the following

Lemma 3.8. *(1) If α is the differential of a function, then $\alpha \mid v = 0$ for every divergence-free field v on M tangent to ∂M.*

(2) Conversely, if $\alpha \mid v = 0$ for every divergence-free field v on M tangent to ∂M, then α is the differential of a function on M.

(3) If for a given v, one has $\alpha \mid v = 0$ for α the differential of every function on M, then the vector field v is divergence-free and tangent to the boundary ∂M.

The proof of this lemma in a more general situation of a (not necessarily simply connected) manifold of arbitrary dimension is given in Section 8. This lemma

manifests the formal identification of the space $\mathfrak{g}^*$ dual to the Lie algebra of divergence-free vector fields in M tangent to the boundary ∂M with the quotient space $\Omega^1(M)/d\Omega^0(M)$ (of all 1-forms on M modulo full differentials). Below we use the identification in this "formal" sense. In order to make precise sense of the discussed duality according to the standards of functional analysis, one has to specify a topology in one of the spaces and to complete the other accordingly. Here we will not fix the completions, and we will regard the elements of both of $\mathfrak{g}$ and $\mathfrak{g}^*$ as smooth functions (fields, forms) unless otherwise specified.

Lemma 3.9. *Let M be a two-dimensional simply connected domain with an area form μ. Then the map $\alpha \mapsto f$ given by*

$$d\alpha = f\mu$$

(where μ is the fixed area element and α is a 1-form in M) defines a natural isomorphism of the space $\Omega^1/d\Omega^0 = \mathfrak{g}^*$, dual to the Lie algebra $\mathfrak{g}$ of divergence-free fields in M (tangent to its boundary ∂M), and the space of functions f in M.

Proof. Adding a full differential to α does not change the function f. Hence we have constructed a map of $\mathfrak{g}^* = \Omega^1/d\Omega^0$ to the space of functions f on M. Since M is simply connected, every function f is the image of a certain closed 1-form α, determined modulo the differential of a function.   □

Theorem 3.10. *The coadjoint representation of the group $S\operatorname{Diff}(M)$ in $\mathfrak{g}^*$ is the natural action of diffeomorphisms, preserving the area element of M, on functions on M.*

Proof. It follows from the fact that all our identifications are natural, i.e., invariant with respect to transformations belonging to $S\operatorname{Diff}(M)$.   □

3.C. The Lie algebra of divergence-free vector fields and its dual in arbitrary dimension

Let $G = S\operatorname{Diff}(M^n)$ be the group of diffeomorphisms preserving a volume element μ on a manifold M with boundary ∂M (in general, M is of any dimension n and multiconnected, but it is assumed to be compact).

The commutator $[v, w]$ (or, $L_v w$) in the corresponding Lie algebra of divergence-free vector fields on M tangent to ∂M is given by minus their Poisson bracket: $[v, w] = -\{v, w\}$; see Example 2.9.

Theorem 3.11 (see, e.g., [M-W]). *The Lie algebra $\mathfrak{g}$ of the group $G = S\operatorname{Diff}(M)$ is naturally identified with the space of closed differential $(n-1)$-forms on M vanishing on ∂M. Namely, a divergence-free field v is associated to the $(n-1)$-form $\omega_v = i_v\mu$. The dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$ is $\Omega^1(M)/d\Omega^0(M)$. The adjoint*

and coadjoint representations are the standard actions of the diffeomorphisms on the corresponding differential forms.

The proof is given in Section 8.

Example 3.12. Let M be a three-dimensional simply connected domain with boundary. Consider the group $S\,\mathrm{Diff}(M)$ of diffeomorphisms preserving the volume element μ (for simply connected M, this group coincides with the group of so-called *exact diffeomorphisms*; see Section 8). Its Lie algebra $\mathfrak{g}$ consists of divergence-free vector fields in M tangent to the boundary ∂M. In the simply connected case the dual space $\mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$ can be identified with *all closed 2-forms* in M by taking the differential of the forms from $\Omega^1(M)$.

We will see below that the vorticity field for a flow with velocity $v \in \mathfrak{g}$ in M is to be regarded as an element of the dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$. The reason is that every 2-form that is the differential of a 1-form corresponds to a certain vorticity field.

On a non-simply connected manifold, the space $\mathfrak{g}^*$ is somewhat bigger than the set of vorticities. In the latter case the physical meaning of the space $\mathfrak{g}^*$, dual to the Lie algebra $\mathfrak{g}$, is the space of circulations over all closed curves. The vorticity field determines the circulations of the initial velocity field over all curves that *are boundaries* of two-dimensional surfaces lying in the domain of the flow. Besides the above, a vector from $\mathfrak{g}^*$ keeps the information about circulation over all other closed curves that *are not boundaries* of anything.

§4. Left-invariant metrics and a rigid body for an arbitrary group

A Riemannian metric on a Lie group G is *left-invariant* if it is preserved under every left shift L_g. The left-invariant metric is defined uniquely by its restriction to the tangent space to the group at the unity, i.e., by a quadratic form on the Lie algebra $\mathfrak{g}$ of the group.

Let $A : \mathfrak{g} \to \mathfrak{g}^*$ be a symmetric positive definite operator that defines the inner product

$$\langle \xi, \eta \rangle = (A\xi, \eta) = (A\eta, \xi)$$

for any ξ, η in $\mathfrak{g}$. (Here the round brackets stand for the pairing of elements of the dual spaces $\mathfrak{g}$ and $\mathfrak{g}^*$.) The positive-definiteness of the quadratic form is not very essential, but in many applications, such as motion of a rigid body or hydrodynamics, the corresponding quadratic form plays the role of kinetic energy.

Definition 4.1. The operator A is called the *inertia operator*.

Define the symmetric linear operator $A_g : T_g G \to T_g^* G$ at every point g of the group G by means of the left translations from g to the unity:

$$A_g \xi = L_g^{*-1} A L_{g^{-1}*} \xi.$$

At every point g, we obtain the inner product

$$\langle \xi, \eta \rangle_g = (A_g \xi, \eta) = (A_g \eta, \xi) = \langle \eta, \xi \rangle_g,$$

where $\xi, \eta \in T_g G$. This product determines the left-invariant Riemannian metric on G. Thus we obtain the commutative diagram in Fig. 5.
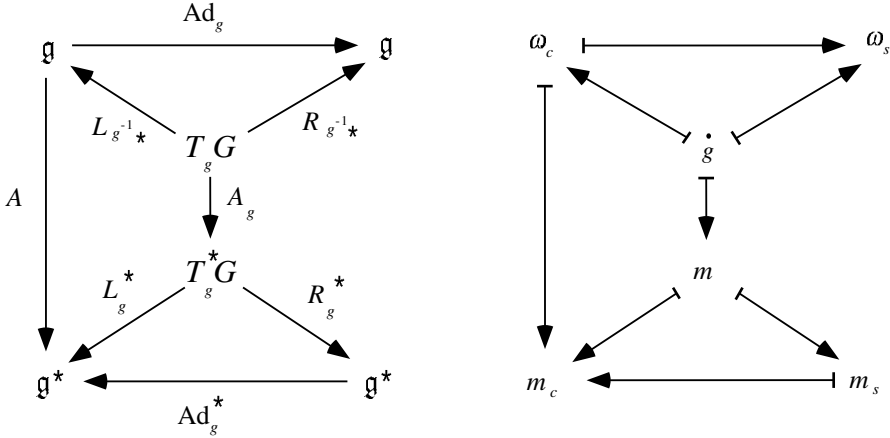
FIGURE 5.   Diagram of the operators in a Lie algebra and in its dual.

Example 4.2. For a classical rigid body with a fixed point, the configuration space is the group $G = SO(3)$ of rotations of three-dimensional Euclidean space. A motion of the body is described by a curve $t \mapsto g(t)$ in the group. The Lie algebra $\mathfrak{g}$ of the group G is the three-dimensional space of angular velocities of all possible rotations. The commutator in this Lie algebra is the usual cross product.

A rotation velocity $\dot{g}(t)$ of the body is a tangent vector to the group at the point $g(t)$. By translating it to the identity via left or right shifts, we obtain two elements of the Lie algebra $\mathfrak{g}$.

Definition 4.3. The result of the left translation is called the *angular velocity in the body* (and is denoted by ω_c with c for "corps" = body), while the result of the right translation is the *spatial angular velocity* (denoted by ω_s),

$$\omega_c = L_{g^{-1}*} \dot{g} \in \mathfrak{g}, \qquad \omega_s = R_{g^{-1}*} \dot{g} \in \mathfrak{g}.$$

Note that $\omega_s = \mathrm{Ad}_g \, \omega_c$.

The space $\mathfrak{g}^*$, dual to the Lie algebra $\mathfrak{g}$, is called the *space of angular momenta*. The symmetric operator $A : \mathfrak{g} \to \mathfrak{g}^*$ is the *operator* (or tensor) *of inertia momentum*. It is related to the kinetic energy E by the formula

$$E = \frac{1}{2}\langle \dot{g}, \dot{g} \rangle_g = \frac{1}{2}\langle \omega_c, \omega_c \rangle = \frac{1}{2}(A\omega_c, \omega_c) = \frac{1}{2}(A_g \dot{g}, \dot{g}).$$

The image m of the velocity vector $\dot{g}$ under the action of the operator A_g belongs to the space T_g^*G. This vector can be carried to the cotangent space to the group G at the identity by both left or right translations. The vectors

$$m_c = L_g^* m \in \mathfrak{g}^*, \qquad m_s = R_g^* m \in \mathfrak{g}^*$$

are called the vector of the *angular momentum relative to the body* (m_c) and that of the *angular momentum relative to the space* (or *spatial angular momentum*, m_s). Note that $m_c = \mathrm{Ad}_g^* m_s$.

The kinetic energy is given by the formula

$$E = \frac{1}{2}(m_c, \omega_c) = \frac{1}{2}(m, \dot{g})$$

in terms of momentum and angular velocity. The quadratic form E defines a left-invariant Riemannian metric on the group. According to the least action principle, inertia motions of a rigid body with a fixed point are geodesics on the group $G = SO(3)$ equipped with this left-invariant Riemannian metric. (Note that in the case $SO(3)$ of the motion in three-dimensional space, the inertia operators of genuine rigid bodies form an open set in the space of all symmetric operators $A : \mathfrak{g} \to \mathfrak{g}^*$ (some triangle inequality should be satisfied).)

Similarly, in the general situation of a left-invariant metric on an arbitrary Lie group G, we consider four vectors moving in the spaces $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively:

$$\omega_c(t) \in \mathfrak{g}, \quad \omega_s(t) \in \mathfrak{g}, \quad m_c(t) \in \mathfrak{g}^*, \quad m_s(t) \in \mathfrak{g}^*.$$

They are called the vectors of angular velocity and momentum in the body and in space.

L. Euler [Eul] found the differential equations that these moving vectors satisfy:

Theorem 4.4 (First Euler Theorem). *The vector of spatial angular momentum is preserved under motion:*

$$\frac{dm_s}{dt} = 0.$$

Theorem 4.5 (Second Euler Theorem). *The vector of angular momentum relative to the body obeys the Euler equation*

$$(4.1) \qquad\qquad \frac{dm_c}{dt} = \mathrm{ad}_{\omega_c}^* m_c.$$

Remark 4.6. The vector $\omega_c = A^{-1} m_c$ is linearly expressed in terms of m_c. Therefore, the Euler equation defines a quadratic vector field in $\mathfrak{g}^*$, and its flow describes the evolution of the vector m_c. The latter evolution of the momentum vector depends only on the position of the momentum vector in the body, but not in the ambient space.

In other words, the geodesic flow in the phase manifold T^*G is fibered over the flow of the Euler equation in the space $\mathfrak{g}^*$, whose dimension is one half that of T^*G.

Proofs. Euler proved his theorems for the case of $G = SO(3)$, but the proofs are almost literally applicable to the general case. Namely, the First Euler Theorem is the conservation law implied by the energy symmetry with respect to left translations. The Second Euler Theorem is a formal corollary of the first and of the identity

$$(4.2) \qquad\qquad m_c(t) = \mathrm{Ad}^*_{g(t)} \, m_s.$$

Differentiating the left- and right-hand sides of the identity in t at $t = 0$ (and assuming that $g(0) = e$), we obtain the Euler equation (4.1) for this case. The left-invariance of the metric implies that the right-hand side depends solely on m_c, but not on $g(t)$, and therefore the equation is satisfied for every $g(t)$. □

Remark 4.7. The Euler equation (4.1) for a rigid body in $\mathbb{R}^3$ is $\dot{m} = m \times \omega$ for the angular momentum $m = A\omega$. For $A = \mathrm{diag}(I_1, I_2, I_3)$ one has

$$\begin{cases} \dot{m}_1 = \gamma_{23} m_2 m_3, \\ \dot{m}_2 = \gamma_{31} m_3 m_1, \\ \dot{m}_3 = \gamma_{12} m_1 m_2, \end{cases}$$

where $\gamma_{ij} = I_j^{-1} - I_i^{-1}$. The principal inertia momenta I_i satisfy the triangle inequality $|I_i - I_j| \leq I_k$.

The relation (4.2) and the First Euler Theorem imply the following

Theorem 4.8. *Each solution $m_c(t)$ of the Euler equation belongs to the same coadjoint orbit for all t. In other words, the group coadjoint orbits are invariant submanifolds for the flow of the Euler equation in the dual space $\mathfrak{g}^*$ to the Lie algebra.*

The isomorphism $A^{-1} : \mathfrak{g}^* \to \mathfrak{g}$ allows one to rewrite the Euler equation on the Lie algebra as an evolution law on the vector $\omega_c = A^{-1} m_c$. The result is as follows.

Theorem 4.9. *The vector of angular velocity in the body obeys the following equation with quadratic right-hand side:*

$$\frac{d\omega_c}{dt} = B(\omega_c, \omega_c),$$

where the bilinear (nonsymmetric) form $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is defined by

$$(4.3) \qquad\qquad \langle [a, b], c \rangle = \langle B(c, a), b \rangle$$

for every a, b, c in $\mathfrak{g}$. Here, $[\cdot, \cdot]$ is the commutator in the Lie algebra $\mathfrak{g}$, and $\langle \cdot, \cdot \rangle$ is the inner product in the space $\mathfrak{g}$.

Remark 4.10. The operation B is bilinear, and for a fixed first argument, it is skew symmetric with respect to the second argument:

$$\langle B(c, a), b \rangle + \langle B(c, b), a \rangle = 0.$$

The operator B is the image of the operator of the algebra coadjoint representation under the isomorphism of $\mathfrak{g}$ and $\mathfrak{g}^*$ defined by the inertia operator A.

Proof of Theorem 4.9. For each $b \in \mathfrak{g}$, we have

$$\langle \frac{d\omega_c}{dt}, b \rangle = \langle A^{-1}\frac{dm_c}{dt}, b \rangle = \frac{dm_c}{dt} \mid b,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in the Lie algebra, and $. \mid .$ stands for the pairing of elements from $\mathfrak{g}$ and $\mathfrak{g}^*$. By virtue of the Euler equation,

$$\frac{dm_c}{dt} \mid b = (\mathrm{ad}^*_{\omega_c} m_c) \mid b = m_c \mid \mathrm{ad}_{\omega_c} b = m_c \mid [\omega_c, b].$$

By definition of the inner product,

$$m_c \mid [\omega_c, b] = (A\omega_c) \mid [\omega_c, b] = \langle [\omega_c, b], \omega_c \rangle.$$

The definition of the operation B allows one to rewrite it as

$$\langle [\omega_c, b], \omega_c \rangle = \langle B(\omega_c, \omega_c), b \rangle.$$

Thus, for each b we finally have

$$\langle \frac{d\omega_c}{dt}, b \rangle = \langle B(\omega_c, \omega_c), b \rangle,$$

which proves Theorem 4.9. □

Remark 4.11. Consider the motion of a three-dimensional rigid body. The Euler equation (4.1) describes the evolution of the momentum vector in the three-dimensional space $\mathfrak{so}(3, \mathbb{R})^*$. Each solution $m_c(t)$ of the Euler equation belongs to the intersection of the coadjoint orbits (which are spheres centered at the origin) with the the energy levels; see Fig. 6. The kinetic energy is a quadratic first integral on the dual space, and its level surfaces are ellipsoids $\langle A^{-1}m_c, m_c \rangle = \mathrm{const}$.

The dynamics of an n-dimensional rigid body is naturally associated to the group $SO(n, \mathbb{R})$. The trajectories of the corresponding Euler equation are no longer determined by the intersections of the coadjoint orbits of this group with the energy levels (see Section VI.1.B).

In the next section we will apply the Euler theorems to the (infinite-dimensional) group of volume-preserving diffeomorphisms [Arn4, 16]. Note that the analogy between the Euler equations for ideal hydrodynamics and for a rigid body was pointed out by Moreau in [Mor1].
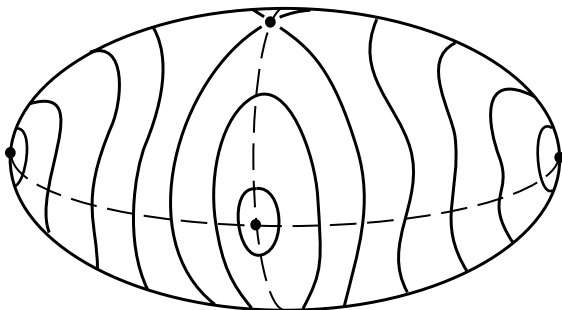
FIGURE 6.   Trajectories of the Euler equation on an energy level surface.

§5. Applications to hydrodynamics

According to the principle of least action, motions of an ideal (incompressible, inviscid) fluid in a Riemannian manifold M are geodesics of a *right*-invariant metric on the Lie group $S\,\mathrm{Diff}\,(M)$. Such a metric is defined by the quadratic form E (E being the kinetic energy) on the Lie algebra of divergence-free vector fields:

$$E = \frac{1}{2} \iint\limits_{M} v^2 \mu,$$

where μ is a volume element on M, and v^2 is the square of Riemannian length of a vector tangent to M.

Remark 5.1. To carry out the passage from left-invariant metrics to right-invariant ones, it suffices to change the sign of the commutator $[\,,\,]$ (as well as of all operators linearly depending on it: $\mathrm{ad}_v \cdot = [v, \cdot]$, ad_v^*, B) in all the formulas. Indeed, the Lie group G remains a group after the change of the product $(g, h) \mapsto gh$ to $(g, h) \mapsto g * h = hg$.

The Lie algebra commutator changes sign under this transform, while a left-invariant metric becomes right invariant. Of course, left translations with respect to the old group operation become right translations for the new one. Therefore, for right-invariant metrics the result of the *right* translation of a momentum vector to the dual Lie algebra is preserved in time, while the left translation of the momentum obeys the Euler equation.

In hydrodynamics the metric on the group is right invariant. Hence, from the general results of the preceding section we obtain the (Euler) equations of motion of an ideal fluid (on a Riemannian manifold of arbitrary dimension), as well as the conservation laws for them.

The Euler equations on a flow *velocity field* in the domain M are the result of a *right* shift to the *Lie algebra* $\mathfrak{g} = S\,\mathrm{Vect}(M)$ of divergence-free vector fields on M (see Theorem 4.8, with the change of the left shift to the right one). The right

invariance of the metric results in the following form of the Euler equation:

$$\dot{v} = -B(v, v),$$

where the operation B on the Lie algebra $\mathfrak{g}$ is defined by (4.3). Its equivalent form is the *Euler–Helmholtz equation* on the *vorticity field*, i.e., equation (4.1) with the opposite sign for right shifts of *momentum* to the *dual space* $\mathfrak{g}^*$ of the Lie algebra.

Example 5.2. Consider the Lie algebra $\mathfrak{g} = S\,\mathrm{Vect}(M)$ of divergence-free vector fields on a simply connected domain M, tangent to ∂M, with the commutator $[\cdot, \cdot] = -\{\cdot, \cdot\}$ being minus the Poisson bracket. Below we show that the operation B for the Euler equation on this Lie algebra has the form

(5.1) $$B(c, a) = \mathrm{curl}\, c \times a + \mathrm{grad}\, p,$$

where $\times$ is the cross product and p is a function on M, determined uniquely (modulo an additive constant) by the condition $B \in \mathfrak{g}$ (i.e., by the conditions div $B = 0$ and tangency of $B(c, a)$ to ∂M). Hence, the *Euler equation for three-dimensional ideal hydrodynamics* is the evolution

(5.2) $$\frac{\partial v}{\partial t} = v \times \mathrm{curl}\, v - \mathrm{grad}\, p$$

of a divergence-free vector field v in $M \subset \mathbb{R}^3$ tangent to ∂M.

The *vortex* (or the *Euler–Helmholtz*) *equation* is as follows:

(5.3) $$\frac{\partial \omega}{\partial t} = -\{v, \omega\}, \qquad \omega = \mathrm{curl}\, v.$$

Proof. By definition of the operation B,

$$\langle B(c, a), b \rangle = \langle [a, b], c \rangle,$$

where $[a, b]$ is the commutator in the Lie algebra $S\,\mathrm{Vect}(M)$ (equal to $-\{a, b\}$ in terms of the Poisson bracket). Since all fields are divergence free, we have

$$\langle [a, b], c \rangle = \langle \mathrm{curl}(a \times b), c \rangle = \langle a \times b, \mathrm{curl}\, c \rangle = \langle (\mathrm{curl}\, c) \times a, b \rangle.$$

Thus, $\mathrm{curl}\, c \times a$ gives the explicit form of the operation B, modulo a gradient term (since div $b = 0$).

The vortex equation is obtained from the Euler equation on the velocity field by taking curl of both sides. $\qquad \square$

Formula (5.1) holds in a more general situation of a Riemannian three-dimensional manifold M with boundary. Moreover, for a manifold of arbitrary dimension, one can still make sense of this formula by specifying the definition of the cross product.

Theorem 5.3. *The operation $B(v, v)$ for a divergence-free vector field v on a Riemannian manifold M of any dimension is*

$$B(v, v) = \nabla_v v + \mathrm{grad}\, p.$$

Here $\nabla_v v$ is the vector field on M, that is the covariant derivative of the field v along itself in the Riemannian connection on M related to the chosen Riemannian metric, and p is determined modulo a constant by the same conditions as above.

We postpone the proof of this theorem until the discussion of covariant derivative in Section IV.1. The proof is based on the following simple interpretation of the inertia operator for hydrodynamics. As we discussed above, the Lie algebra of divergence-free vector fields and its dual space can be defined as soon as the manifold is equipped with a volume form. The inertia operator requires an additional structure, a *Riemannian metric on the manifold*, similar to fixing an inertia ellipsoid for a rigid body.

Theorem 5.4. *The inertia operator for ideal hydrodynamics on a Riemannian manifold takes a velocity vector field to the 1-form whose value on an arbitrary vector equals the Riemannian inner product of the latter vector with the velocity vector at that point (the obtained 1-form is regarded modulo the differentials of functions).*

See the proof in Section 7 (Theorem 7.19).

In the case of hydrodynamics, the invariance of coadjoint orbits with respect to the Euler dynamics (Theorem 4.8) takes the form of Helmholtz's classical theorem on vorticity conservation.

Theorem 5.5. *The circulation of any velocity field over each closed curve is equal to the circulation of this velocity field, as it changes according to the Euler equation, over the curve transported by the fluid flow.*

Proof. Consider an element of the Lie algebra $S\operatorname{Vect}(M)$ corresponding to a "narrow current" that flows along the chosen curve and has unit flux across a transverse to the curve. Under the adjoint representation (i.e., action of a volume-preserving diffeomorphism), this element is taken to a similar "narrow current" along the transported curve.

The pairing of a vector of the dual Lie algebra with the chosen element in the Lie algebra itself is the integral of the corresponding 1-form along the curve (note that although an element of the dual space is a 1-form modulo any function differential, its integral over a closed curve is well-defined). By Theorem 5.4, the latter pairing is the circulation of the velocity field along our curve. $\square$

The above theorem implies that the velocity fields (parametrized by time t) that constitute one solution of the Euler equation are isovorticed; i.e., the vorticity of the field at any given moment of time t is transported to the vorticity at any other moment by a diffeomorphism preserving the volume element.

Remark 5.6. Isovorticity, i.e., the condition on phase points to belong to the same coadjoint orbit, imposes constraints that differ drastically in two- and three-

dimensional cases. For a two-dimensional fluid the coadjoint orbits are distinguished by the values of the first integrals, such as vorticity momenta. In the three-dimensional case the orbit geometry is much more subtle.

Owing to this difference in the geometry of coadjoint orbits, the foundation of three-dimensional hydrodynamics encounters serious difficulties. Meanwhile, in the hydrodynamics of a two-dimensional fluid, the existence and uniqueness of global solutions have been proved [Yu1], and the proofs use heavily the first integrals of the Euler equation, which are invariant on the coadjoint orbits.

Definition 5.7. Given a velocity vector field, consider the 1-form that is the (pointwise) Riemannian inner product with the velocity field. Its differential is called the *vorticity form*.

Example 5.8. On the Euclidean plane (x, y) this 2-form is $\omega \, dx \wedge dy$, where ω is a function. The function ω, also called the vorticity of a two-dimensional flow, is related to the stream function ψ by the identity $\omega = \Delta \psi$.

In three-dimensional Euclidean space this is the 2-form corresponding to the vorticity vector field curl v. Its value on a pair of vectors equals their mixed product with curl v.

Definition 5.9. The *vorticity vector field* of an incompressible flow on a three-dimensional Riemannian manifold is defined as the vector field ξ associated to the vorticity 2-form ω according to the formula

$$\omega = i_\xi \mu,$$

where μ is the volume element. In other words, the vorticity vector ξ is defined at each point by the condition

$$(5.4) \qquad\qquad \mu(\xi, a, b) = \omega(a, b)$$

for any pair of vectors a, b attached at that point. One has to note that the construction of the field ξ does not use any coordinates or metric but only the volume element μ and the 2-form ω.

Remark 5.10. On a manifold of an arbitrary dimension n the vorticity is not a vector field but an $(n-2)$-polyvector field (k-*polyvector*, or k-vector, is a polylinear skew-symmetric function of k cotangent vectors, i.e., of k 1-forms at the point). For instance, for $n = 2$, one obtains the 0-polyvector, that is, a scalar. Such a scalar is the vorticity function ω of a two-dimensional flow in the example above. From Theorem 5.5 follows

Corollary 5.11. *The vorticity field is frozen into the incompressible fluid.*

Indeed, by virtue of Theorem 5.5, the vorticity 2-form ω is transported by the flow, since it is the differential of the 1-form "inner product with v," which is

transported. The volume 3-form μ is also transported by the flow (since the fluid is incompressible).

In turn, the vorticity field ξ is defined by the forms ω and μ in an invariant way (without the use of a Riemannian metric) by formula (5.4). Therefore, this field is "frozen"; i.e., it is transported by fluid particles just as if the field arrows were drawn on the particles themselves: A stretching of a particle in any direction implies the stretching of the field in the same direction.

Remark 5.12. Consider any diffeomorphism preserving the volume element (but *a priori* not related to any fluid flow). If such a diffeomorphism takes a vorticity 2-form ω_1 into a vorticity 2-form ω_2, then it transports the vorticity field ξ_1 corresponding to the first form to the vorticity field ξ_2 corresponding to the second.

If, however, one starts with a velocity field and then associates to it the corresponding vorticity field, the vorticity transported by an arbitrary diffeomorphism is not, in general, the vorticity for the velocity field obtained from the initial velocity by the diffeomorphism action. Theorem 5.5 states that the coincidence holds for the family of diffeomorphisms that is the Euler flow of an incompressible fluid with a given initial velocity field. In other words, the momentary velocity fields in the same Euler flow are isovorticed.

Corollary 5.13. *The vorticity trajectories are transported by an Eulerian fluid motion on a three-dimensional Riemannian manifold.*

In particular, every "vorticity tube" (i.e., a pencil of vorticity lines) is carried along by the flow. The Helmholtz theorem is closely related to this geometric corollary but is somewhat stronger (especially in the non-simply connected case).

Remark 5.14. In the two-dimensional case, the isovorticity of velocity vector fields means that the vorticity function ω is transported by the fluid flow: A point where the vorticity $\Delta\psi$ was equal to ω at the initial moment is taken to a point with the same vorticity value at any other moment of time. In particular, all vorticity momenta

$$I_k = \iint (\Delta\psi)^k \, dx \, dy$$

are preserved, and so are the areas of the sets of smaller vorticity values:

$$S(c) = \iint\limits_{\Delta\psi \le c} dx \, dy$$

(see, e.g., [Ob]). The same holds in a non-simply connected situation.

The conservation laws provided by the Helmholtz theorem are a bit stronger than the conservation of all the momenta, even if the two-dimensional manifold is simply connected. Namely, one claims that the whole "tree" of the vorticity function $\omega = \Delta\psi$ (that is, the space of components of the level sets; see Fig. 7) is preserved, as well as the vorticity function ω, along with the measure on this tree.

(The latter measure associates to every segment on the tree the total area of the corresponding vorticity levels.) Apparently, it is the complete set of invariants of typical coadjoint orbits (say, for $S\,\mathrm{Diff}(S^2)$).
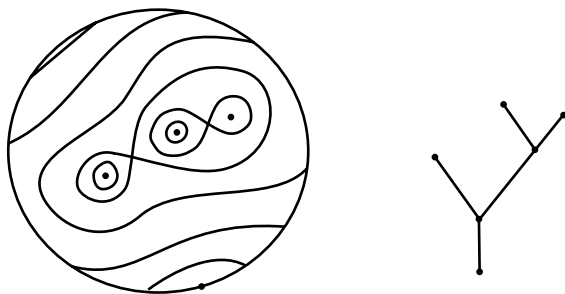
FIGURE 7.   The "tree" of a vorticity function on a sphere.

It would be interesting to describe possible graphs for functions on non-simply connected manifolds, for instance, on a torus.

Applications of the Euler theorems to the hydrodynamics on manifolds of higher dimension are described in Section 7.

Remark 5.15.[1] Though, as mentioned at the beginning, we are not dealing here with the existence and uniqueness theorems for the Euler equation of an ideal incompressible fluid, it is a very subtle question that has attracted considerable interest in the literature (see, e.g., [Chm, Gé, Yu3] for a survey). Local in time existence and uniqueness theorems of the classical solution of the basic initial boundary value problem for the two- and three-dimensional Euler equation were obtained in a series of papers by N. Gunter and L. Lichtenstein. W. Wolibner proved the global solvability for the 2-D problem for the classical solutions (see [Ka] for the modern form of the result and generalizations). The global existence theorem in two-dimensional Eulerian hydrodynamics was proved by V. Yudovich [Yu1] for flows with vorticity in the space L^p for any given $p > 1$. For the uniqueness theorem on flows with essentially bounded vorticity and its generalizations see [Yu1, 3]; the nonuniqueness of weak solutions of the Euler equations is discussed in [Shn7].

If instead of an ideal fluid we consider a viscous incompressible one, its motion is described by the Navier–Stokes equation, being the Euler equation with an additional diffusion term; see Section 12. The local in time results (existence and uniqueness) for classical solutions of the Navier–Stokes equation were obtained by Lichtenstein, Odqvist, and Oseen (see the references in [Lad]). The global (in time and in "everything," i.e., the domain, initial field, and viscosity) existence

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[1]We are grateful to V. Yudovich for consulting us on the history of this question.

of generalized solutions was proved by J. Leray (in the 1930s) and E. Hopf (in 1950/51). Uniqueness for this wide class of solutions is still unknown. The global existence and uniqueness theorems of generalized and classical solutions of the 2-D Navier–Stokes equation were proved by Ladyzhenskaya and her successors (see [Lad]).

§6. Hamiltonian structure for the Euler equations

Recall the coadjoint representation of an arbitrary Lie group G. It turns out that the coadjoint orbits are always even-dimensional. The reason is that such an orbit is endowed with a natural symplectic structure (i.e., a closed nondegenerate 2-form). This structure, called the Kirillov (Berezin, Kostant) form (see [Ki1, Ber, Kos]), was essentially discovered by S. Lie [Lie].

The Euler equations in the dual space to a Lie algebra are Hamiltonian equations on each coadjoint orbit [Arn5]. Now the kinetic energy plays the role of the corresponding Hamiltonian function. We start with the following brief reminder.

Let (M, ω) be a symplectic manifold, i.e., a manifold M equipped with a closed nondegenerate differential 2-form. Recall that a Hamiltonian function H defines a Hamiltonian field v on M by the condition

$$(6.1) \qquad\qquad i_v \omega = -dH.$$

In other words, the field v is the skew gradient of the function $H : M \to \mathbb{R}$ defined by the relation $-dH(\xi) = \omega(v, \xi)$ for every ξ (the ordinary gradient of a function is defined by the condition $dH(\xi) = \langle \operatorname{grad} H, \xi \rangle$ for every ξ, where $\langle \cdot, \cdot \rangle$ is an inner product on a Riemannian manifold M). The value of the skew-symmetric 2-form ω on a pair of vectors (v and ξ in the case at hand) is called their skew-symmetric product. The following theorem is well known (see, e.g., [Arn16]).

Theorem 6.1. *The phase flow of the Hamiltonian field v preserves the symplectic form ω and the Hamiltonian function H.*

Now assume that M is a coadjoint orbit of a Lie group G. The manifold M is embedded into the dual space $\mathfrak{g}^*$ of the corresponding Lie algebra. The tangent space to the orbit M at every point is spanned by the velocity vectors of the coadjoint representation corresponding to arbitrary velocities with which an element of the group G leaves the unity. In our notation (see Section 4), these vectors attached at a point $m \in M$ and tangent to M have the form

$$\xi = \operatorname{ad}_a^* m, \qquad m \in \mathfrak{g}^*, a \in \mathfrak{g}.$$

Now consider two such vectors, corresponding to two "angular velocities" (i.e., elements of the Lie algebra $\mathfrak{g}$)

$$\xi = \operatorname{ad}_a^* m, \ \eta = \operatorname{ad}_b^* m, \qquad a, b \in \mathfrak{g}.$$

One can combine these two vectors and one element of the dual space to get the number

$$(6.2) \qquad \omega(\xi, \eta) := (m, [a, b]),$$

where the square brackets denote the commutator in the Lie algebra, and the round ones stand for the natural pairing between the dual spaces $\mathfrak{g}^*$ and $\mathfrak{g}$. One easily proves the following result.

Theorem 6.2. *The value $\omega(\xi, \eta)$ depends on the vectors ξ and η tangent to M at m, but not on a particular choice of the "angular velocities" a and b used in the definition. The skew-symmetric form ω on M is closed and nondegenerate.*

This form defines the symplectic structure on the coadjoint orbit. It is invariant under the coadjoint representation (which follows from its definition).

Example 6.3. For $G = SO(3)$ the coadjoint orbits are all spheres centered at the origin and the origin itself (note that all the dimensions are even!). Symplectic structures are the area elements invariant with respect to rotations of the spheres. The areas are normalized by the following condition: $\iint \omega$ is proportional to the sphere radius.

Formula (6.2) defines the symplectic structures on all coadjoint orbits at once. These symplectic structures are related in such a way that they equip the entire dual Lie algebra with a more general structure called the Poisson structure.

Definition 6.4. A *Poisson structure* on a manifold is an operation $\{\cdot, \cdot\}$ that associates to a pair of smooth functions on the manifold a third one (their *Poisson bracket*) such that the operation is bilinear and skew-symmetric, and it satisfies the Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

and the Leibniz identity

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

A manifold equipped with a Poisson structure is called a *Poisson manifold*.

The Leibniz identity means that for a fixed first argument, the operation $\{\cdot, \cdot\}$ is the differentiation of the second argument along some vector field.

Example 6.5. Consider all smooth functions on the dual space $\mathfrak{g}^*$ of a finite-dimensional Lie algebra. Define the Poisson bracket on this space by

$$(6.3) \qquad \{f, g\}(m) := (m, [df, dg]) \qquad \text{for} \quad m \in \mathfrak{g}^*, \; f, g \in C^\infty(\mathfrak{g}^*),$$

where the differentials df and dg are taken at the same point m. Note that the differential of f at each point $m \in \mathfrak{g}^*$ is an element of the Lie algebra $\mathfrak{g}$ itself.

Hence, the commutator $[df, dg]$ at every point is also a vector of this Lie algebra. The value of the linear function m evaluated at the latter vector, appearing on the right-hand side of the above formula, is, generally speaking, a nonlinear function of m, the Poisson bracket of the pair of functions f and g.

Let $x_1, \ldots, x_n$ be coordinates in the dual space to an n-dimensional Lie algebra. Then formula (6.3) assumes the form

$$\{f, g\} = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} [x_i, x_j].$$

Here the vectors x_i form a basis of the Lie algebra $\mathfrak{g}$ itself. Their commutators lie in the Lie algebra as well, and therefore they are (linear) functions on the dual space $\mathfrak{g}^*$.

Definition 6.6. The operation defined above is called the *natural Lie–Poisson structure* on the dual space to a Lie algebra.

One readily verifies (see, e.g., [We]) the following

Theorem 6.7. *This is indeed a Poisson structure.*

Remark 6.8. In fact, Poisson structures, in a somewhat more general situation, were introduced by Jacobi in "Lectures on dynamics" [Jac] while analyzing the structure of the ring of first integrals for a given Hamiltonian vector field. The Jacobi theory is more algebraic than topological, and it defines the Poisson structures on more general sets, similar to varieties of algebraic geometry rather than on the manifolds of topologists. Generally speaking, those sets are not Hausdorff. The modern definition was introduced by A. Weinstein [We], after the works of Lichnerowicz and Kirillov.

Definition 6.9. The *Hamiltonian field* of a function H on a manifold equipped with a Poisson structure is the vector field ξ defined by the relation

$$L_\xi f = \{H, f\}$$

for every function f. Here $L_\xi f$ is the derivative of a function f along the vector field ξ_i, in coordinates $L_\xi f = \sum \xi_i \frac{\partial f}{\partial x_i}$.

Example 6.10. The Hamiltonian field ξ of a linear function a on the dual space of a Lie algebra is given by the formula

$$\xi(\cdot) = \mathrm{ad}_a^* \cdot,$$

where a is understood as a vector of the Lie algebra itself.

Indeed, at every point $m \in \mathfrak{g}^*$ and for every function f on $\mathfrak{g}^*$, one has

$$\{a, f\}(m) = (m, [a, df]) = (\mathrm{ad}_a^* m, df) = (L_\xi f)(m).$$

More generally, the Hamiltonian field ξ_H of an arbitrary smooth function H on the dual space g^* is given at a point $m \in g^*$ by the vector

$$\xi_H(m) = \mathrm{ad}^*_{dH}\, m,$$

where the differential dH is taken at the point m and is regarded as a vector of the Lie algebra.

Remark 6.11. The Hamiltonian field $\xi_{\{F,H\}}$ associated to the Poisson bracket of two functions F and H is the Poisson bracket of the Hamiltonian fields ξ_F and ξ_H of these functions:

$$\{\xi_F, \xi_H\} = \xi_{\{F,H\}}.$$

It follows from definitions and the Jacobi identity that

$$L_{\{\xi_F,\xi_H\}} f = L_{\xi_F} L_{\xi_H} f - L_{\xi_H} L_{\xi_F} f = L_{\xi_F}\{H, f\} - L_{\xi_H}\{F, f\}$$
$$= \{F, \{H, f\}\} - \{H, \{F, f\}\} = -\{f, \{F, H\}\} = L_{\xi_{\{F,H\}}} f.$$

Definition 6.12. The *symplectic leaf* of a point on a manifold equipped with a Poisson structure is the set of all points of the manifold that can be reached by paths issuing from the given point, and such that the velocity vectors of the paths are Hamiltonian at every moment (with a Hamiltonian function differentiable in time).

Theorem 6.13. *The symplectic leaf of every point is a smooth even-dimensional manifold. It has a natural symplectic structure defined by $\omega(\xi, \eta) = \{f, g\}(x)$, where ξ and η are vectors of Hamiltonian fields with Hamiltonian functions f and g at the point x.*

In particular, the value $\omega(\xi, \eta)$ does not depend on a particular choice of the functions f and g.

On a Poisson manifold the restriction of a Hamiltonian field to each symplectic leaf coincides with the Hamiltonian field defined by the restriction to this leaf of the same Hamiltonian function.

Example 6.14. Symplectic leaves of the natural Poisson structure in the dual space to a Lie algebra are group coadjoint orbits. The symplectic structures of the leaves defined by this Poisson structure coincide with the natural symplectic structure of coadjoint orbits described above.

Now let $A : \mathfrak{g} \to \mathfrak{g}^*$ be a nondegenerate symmetric inertia operator. Define the *dual quadratic form* on the dual Lie algebra space $\mathfrak{g}^*$ by

$$H(m) := \frac{1}{2}(A^{-1}m, m), \qquad m \in \mathfrak{g}^*.$$

Denote by v the Lie algebra vector $A^{-1}m$. Then $m = Av$, and therefore

$$H(m) = \frac{1}{2}(v, Av)$$

is merely the kinetic energy, corresponding to the "angular velocity" v (or to the velocity field v in hydrodynamics).

Theorem 6.15. *Let the inertia operator A define a left-invariant metric on the Lie group G. Then the Euler velocity field in the dual Lie algebra space $\mathfrak{g}^*$ coincides with the Hamiltonian field, defined by the Hamiltonian function H, with respect to the natural Poisson structure of the dual Lie algebra. Explicitly, the Euler equation on the dual space $\mathfrak{g}^*$ is*

(6.4) $$\dot{m} = \mathrm{ad}^*_{A^{-1}m}\, m, \qquad m \in \mathfrak{g}^*.$$

For the right-invariant metric the corresponding Hamiltonian function is $-H$.

In particular, the Euler field is a Hamiltonian field on every coadjoint orbit, with respect to the natural symplectic structure of the orbit. Its Hamiltonian function is the restriction of the kinetic energy to the orbit.

Proof. The differential dH of the quadratic form $H(m) := \frac{1}{2}(A^{-1}m, m)$ at a point $m \in \mathfrak{g}^*$ is the vector $v = A^{-1}m \in \mathfrak{g}$, which is regarded as a linear functional on the dual space $\mathfrak{g}^*$. According to Example 6.10, the Hamiltonian vector corresponding to this linear functional is $\mathrm{ad}^*_{dH}\, m = \mathrm{ad}^*_{A^{-1}m}\, m$, giving equation (6.4).

The fact that this equation describes the geodesics on the Lie group G with respect to the left-invariant metric is nothing but the Second Euler Theorem.

□

Example 6.16. Consider a solution $v(t)$ of the Euler equation of an ideal fluid in a simply connected domain of three-dimensional Euclidean space.

Let $v_1 = v + \varepsilon u_1 + \cdots$, $v_2 = v + \varepsilon u_2 + \cdots$ be two solutions with isovorticed initial conditions infinitely close to v (ε is small, the dots mean $o(\varepsilon)$ as $\varepsilon \to 0$, all the fields here and below depend on t).

Isovorticity of the fields means that their vorticities can be identified by a diffeomorphism action. For infinitesimally small perturbations of the vorticities $\xi_i = \mathrm{curl}\, u_i$ we obtain at every moment t

$$\xi_i = [a_i, \mathrm{curl}\, v],$$

where a_i are divergence-free fields from our Lie algebra, and $[\cdot, \cdot]$ is minus the Poisson bracket of the divergence-free fields (so that $[a, b] = \mathrm{curl}(a \times b)$).

The fields a_i are not determined uniquely by the perturbations u_i. However, one can define the following invariant of the pair of perturbations that does not depend on this ambiguity.

We associate to the initial field $v(t)$ and to the pair of fields a_i the value

$$\omega = \int_{M^3} ([a_1(t), a_2(t)], v(t)) \, dx \, dy \, dz.$$

The theory above, applied to this example, implies the following result.

Theorem 6.17. *The value of ω is constant (i.e., it does not depend on t), whatever solution v of the Euler equation, and whatever initial fields $a_1(0)$ and $a_2(0)$ defining the perturbations are taken.*

Proof. The Hamiltonian property of the Euler equation on the orbits of isovorticed fields implies that the phase flow of the Euler equation preserves the natural symplectic structure of the set of isovorticed fields. This structure is given by the formula $\omega(\xi_1, \xi_2) = (m, [a_1, a_2])$ for $\xi_i = \mathrm{ad}^*_{a_i} m$, where m is the image in the dual space (to the Lie algebra of the divergence-free fields) of the vector $v(t)$ from the Lie algebra under the map $A : \mathfrak{g} \to \mathfrak{g}^*$ of the inertia operator.

According to Theorem 5.4 on inertia operator, the element m can be identified with the 1-form that is the inner product with the vector field v (the dual space itself is understood as the space of 1-forms modulo function differentials).

The differential of the latter 1-form is the vorticity form corresponding to the vorticity vector field $\xi = \mathrm{curl}\, v$. Then the perturbation of the element m defined by the field a_i is

$$\xi_i = [a_i, \xi],$$

and therefore

$$\omega(\xi_1, \xi_2) = \int_{M^3} ([a_1, a_2], v) \, dx \, dy \, dz.$$

The fact that ω does not depend on t means the invariance of the symplectic structure under the Hamiltonian flow of the Euler equation on the coadjoint orbit of the fields isovorticed with $v(0)$. $\qquad\square$

Remark 6.18. On a two-dimensional simply connected domain the fields a_i are defined by the stream functions ψ_i, and the conserved quantity has the form

$$\omega(\xi_1, \xi_2) = \int_{M^2} \xi \cdot \{\psi_1, \psi_2\} \, dx \, dy.$$

Here $\xi_i = \{\psi_i, \xi\}$, $\xi = \Delta\psi$ is the vorticity function of a nonperturbed flow with the stream function ψ, and $\{f, g\}$ is the Poisson bracket of two functions, $\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$, equal to the Jacobian of the map $(x, y) \mapsto (f(x, y), g(x, y))$.

The Hamiltonian property now implies the following. If the vorticity functions ξ and $\xi + \varepsilon\{\psi_i, \xi\} + \cdots$ evolve in time according to the Euler–Helmholtz equation, then the value of ω is time invariant. (The Hamiltonian formalism can also be exploited in the reverse direction: from known results in hydrodynamics one can deduce some properties of Hamiltonian systems; see [Ko1].)

We discuss the more general case of Riemannian manifolds, instead of just a domain in Euclidean space, and the non-simply connected domains and manifolds in the next section. There is vast literature on the Hamiltonian formalism of the Euler equation on Lie groups and numerous applications (see, e.g., books [GS2, Arn16, MaR, G-P] and papers [M-W, AKh, Ose2]).

§7. Ideal hydrodynamics on Riemannian manifolds

Generalization of hydrodynamics of ideal incompressible fluid to manifolds of high dimension (in particular, to dimensions $n > 3$) is as physically meaningful as consideration of, say, the wave equation for a non-physically large number of space coordinates. The universal setting, however, sheds light on general properties of the Euler equation, as well as on geometry of the groups of diffeomorphisms. In particular, in this section we will treat the three-dimensional hydrodynamics from this universal point of view.

7.A. The Euler hydrodynamic equation on manifolds

Let M^n denote a compact oriented Riemannian manifold with a metric $(\ ,\ )$ and a volume form μ, i.e., a nonvanishing differential form of the highest degree n. We do not assume, in general, any relation of μ to the volume form induced by the metric.

Definition 7.1. The *Euler equation of an ideal incompressible fluid on M* is the following evolution equation on the velocity field v of the fluid on the manifold:

$$(7.1) \qquad \begin{cases} \frac{\partial v}{\partial t} = -(v, \nabla)v - \nabla p, \\ \operatorname{div}_\mu v = 0, \end{cases}$$

where the second equation means that the field v preserves the volume form μ. Here p is a time-dependent function on M that is defined by the condition $\operatorname{div}_\mu(\partial v/\partial t) = 0$ uniquely (up to an additive constant depending on time). The expression $(v, \nabla)v$ denotes the covariant derivative $\nabla_v v$ of the field v along itself for the Riemannian connection on M. In the case of the Euclidean space $M = \mathbb{R}^3$ the Euler equation above assumes the form (5.2).

In the case of a manifold M with boundary, the velocity field is supposed to be tangent to the boundary.

We refer to Section IV.1.B for a definition of the covariant derivative, while for many purposes in this chapter it will be enough to keep in mind the following

Example 7.2. In the case of $M = \mathbb{R}^n$, equipped with the standard metric and volume form, the Euler equation of an ideal incompressible fluid is

$$\frac{\partial v_i}{\partial t} = -\sum_{j=1}^n v_j \frac{\partial v_i}{\partial x_j} - \frac{\partial p}{\partial x_i}$$

on the vector field v obeying $\sum_{j=1}^{n} \partial v_j / \partial x_j = 0$. The covariant derivative in this case is

$$(v, \nabla)v_i = \sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j}.$$

Just as for the two- and three-dimensional cases, the Euler equation (7.1) on a compact n-dimensional manifold M can be regarded as the equation of geodesics on the Lie group $S\,\mathrm{Diff}(M)$ of all diffeomorphisms of the manifold M preserving the volume form μ.

Definition 7.3. The *configuration space* of an ideal incompressible fluid filling the manifold M is the *Lie group* $G = S\,\mathrm{Diff}(M)$ of all diffeomorphisms of M preserving the volume form μ (and belonging to the connected component of the identity). In the case of a manifold with boundary ∂M the group $S\,\mathrm{Diff}(M)$ consists of those volume-preserving diffeomorphisms that leave the boundary ∂M invariant.

The *Lie algebra* $\mathfrak{g} = S\,\mathrm{Vect}(M)$ for this group is formed by divergence-free vector fields on M (tangent to the boundary if $\partial M \neq \emptyset$). The Lie bracket in this algebra is *minus* the Poisson bracket of vector fields.

Now we apply the general algebraic machinery to this Lie algebra. The formulations are in Sections 7.B and 7.C below, and the proofs are in Section 8.

7.B. Dual space to the Lie algebra of divergence-free fields

From now on all objects are supposed to be as smooth as needed. We leave aside the analytic difficulties of the approach to infinite-dimensional groups and algebras, and address the interested reader to [E-M], where the proper formalism of the Sobolev spaces for hydrodynamical data is developed. In the sequel we will need the following notions of the calculus on manifolds.

Definition 7.4. Let $\Omega^k(M)$ (or simply Ω^k) denote the space of smooth differential k-forms on the compact manifold M (possibly with boundary ∂M). The *exterior derivative* operator d increases the degree of the forms by 1, while the *inner derivative* operator i_ξ of substitution of a given vector field ξ into a form as the first argument decreases the degree by 1. These operators are derivations of the algebra of forms in the sense that they satisfy the following identities:

(7.2) $$i_\xi(\alpha \wedge \beta) = (i_\xi \alpha) \wedge \beta + (-1)^k \alpha \wedge (i_\xi \beta),$$

(7.3) $$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta),$$

for any forms $\alpha \in \Omega^k$ and $\beta \in \Omega^l$.

The *Lie derivative* of a differential form ω along a vector field v (tangent to the boundary ∂M if $\partial M \neq \emptyset$) is the time derivative of the form ω transported

(backwards) by diffeomorphisms g_t such that $g_0 = \text{Id}$ and $\dot{g}_0 = v$:

$$(7.4) \qquad\qquad L_v\omega = \frac{d}{dt}\Big|_{t=0} g_t^*\omega,$$

where $g_0(x) \equiv x$, $\frac{d}{dt}\big|_{t=0} g_t(x) = v(x)$. The result $g^*\omega$ of the transport of a k-form ω by a smooth map g is defined by the formula

$$(g^*\omega)(\xi_1, \ldots, \xi_k) = \omega(g_*\xi_1, \ldots, g_*\xi_k),$$

where the linear operator g_* is the differential of the map g.

The *homotopy formula* is the relation

$$L_v = i_v d + d i_v.$$

It is an infinitesimal version of the Leibniz formula: The cylinder boundary is the difference of the top and the bottom, plus the side surface (oriented in the proper way).

Theorem 7.5 (see e.g., [M-W, Nov2, DKN]). *The dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g} = S\,\text{Vect}(M)$ of divergence-free vector fields on M (tangent to ∂M) is naturally isomorphic to the quotient space $\Omega^1/d\Omega^0$ of all differential 1-forms on M, modulo all exact 1-forms (i.e., modulo differentials of all functions) on M.*

The group coadjoint action on the dual Lie algebra $\mathfrak{g}^*$ coincides with the standard action of diffeomorphisms on differential 1-forms:

$$(7.5) \qquad\qquad \text{Ad}_g^* \alpha = g^*\alpha,$$

where 1-forms α and $g^*\alpha$ on M are considered modulo function differentials.

Here we regard $S\,\text{Vect}(M)$ as the Lie algebra of the group of diffeomorphisms of M preserving a fixed volume element. The commutator $\text{ad}_v w = [v, w]$ of two vector fields is thus minus their Poisson bracket. We will prove this theorem in Section 8.

Corollary 7.6. *The algebra coadjoint action by an element $v \in \mathfrak{g}$ on the dual space $\mathfrak{g}^* = \Omega^1/d\Omega^0$ is the Lie derivative of the 1-forms along the vector field v on M:*

$$(7.6) \qquad\qquad \text{ad}_v^* \alpha = L_v\alpha.$$

Here α and $L_v\alpha$ are 1-forms modulo function differentials.

Indeed, the corollary follows directly from the definition of the Lie derivative (7.4). The infinitesimal version of (7.5) for $g \in G$ close enough to the identity (i.e., for the "infinitesimal change of variables" $g = \text{Id} + \epsilon v + o(\epsilon)$ given by a vector field v) determines the coadjoint action of the Lie algebra element $v \in \mathfrak{g} = S\,\text{Vect}(M)$ on the dual space $\mathfrak{g}^*$ as the derivation along the vector field. $\qquad\square$

Definition 7.7. The *pairing* of the spaces $\mathfrak{g}$ and $\mathfrak{g}^*$ is given by the following straightforward formula. Let $[u]$ denote the *coset* of a 1-form u in the quotient

$\Omega^1/d\Omega^0$, i.e., the class of all 1-forms on M of the type $u + df$ for some function f. Then, to evaluate a coset $[u] \in \mathfrak{g}^*$ at a vector field $v \in \mathfrak{g}$, one has to take the integral over M of the pointwise pairing of the vector v and an arbitrary 1-form u from the coset $[u] \in \Omega^1/d\Omega^0$:

$$(7.7) \qquad \langle v, [u] \rangle = \int_M u(v)\mu.$$

(The fact that this integral does not depend on a particular choice of u is proved in Section 8.) Equivalently, one can think of $\Omega^1/d\Omega^0$ as the space dual to the space of all closed $(n-1)$-forms on M:

$$(7.8) \qquad \langle \omega_v, [u] \rangle = \int_M u \wedge \omega_v,$$

where $\omega_v = i_v\mu$ is the closed $(n-1)$-form associated to the divergence-free vector field v.

Remark 7.8. The group coadjoint action is well-defined by the formula (7.5), since the diffeomorphism action commutes with the operation d: If $\alpha' = \alpha + df$, then

$$g^*\alpha' = g^*\alpha + g^*(df) = g^*\alpha + d(g^*f);$$

i.e., the 1-form $g^*\alpha$ and $g^*\alpha'$ define the same coset in the quotient $\Omega^1/d\Omega^0$. Similarly, the Lie derivative acts on the coset of a 1-form u, since the operation L_v commutes with the derivative operator d:

$$L_v[u] = L_v(u + df) = L_vu + dL_vf = [L_vu].$$

As we discussed above, the space $\mathfrak{g}^*$, in the form of the quotient $\Omega^1/d\Omega^0$, is understood only as the regular part of the actual dual space to the Lie algebra $\mathfrak{g} = S\,\mathrm{Vect}(M)$. Notice that the nonregular part of the dual space $\mathfrak{g}^*$ includes many interesting functionals, e.g., singular closed 2-forms supported on submanifolds of codimension 2 (for $n = 2$ such forms are supported in a discrete set of points, while for $n = 3$ the support can be a set of closed curves; see Sections I.11 and VI.3).

Corollary 7.9. *For a simply connected manifold M (or, more generally, for a manifold with trivial first homology group $H_1(M, \mathbb{R}) = 0$), the dual space $\mathfrak{g}^*$ is isomorphic to the space of all exact 2-forms on M.*

Indeed, the kernel of the operator $d : \Omega^1 \to \Omega^2$ contains all closed 1-forms on M. Simply-connectedness of M (or the condition $H_1(M, \mathbb{R}) = 0$) implies that the first cohomology group vanishes: $H^1(M, \mathbb{R}) = 0$; i.e., all closed 1-forms are exact. Thus the quotient $\Omega^1/d\Omega^0$ is isomorphic to the image of d in Ω^2.   □

Definition 7.10. A divergence-free vector field v on M^n (tangent to ∂M) is said to be *exact* if the corresponding closed $(n-1)$-form $\omega_v := i_v\mu$ is the differential of some $(n-2)$-form vanishing on $\partial M : \omega_v = i_v\mu = d\alpha, \alpha|_{\partial M} = 0$.

Example 7.11. On a two-dimensional surface, a field is exact if and only if it possesses a univalued stream function vanishing on the boundary of the surface. The flux of such a field across any closed curve within the surface, as well as the flux across any chord connecting two boundary points, is equal to zero.

On a simply connected manifold of any dimension n every divergence-free vector field is exact. Indeed, due to the identity $H^{n-1}(M) = H^1(M) = 0$, every closed $(n-1)$-form ω_v is exact.

Definition 7.12. A diffeomorphism of a manifold M (preserving the volume element μ and the boundary ∂M) is called *exact* if it can be connected to the identity transformation by a smooth curve g_t (in the space of volume-preserving diffeomorphisms of M) so that the velocity field $\dot{g}_t$ is exact at every moment t. The exact diffeomorphisms constitute the *group of exact diffeomorphisms $S_0 \operatorname{Diff}(M^n)$*.

The latter is a subgroup of the group $S \operatorname{Diff}(M^n)$ of all volume-preserving diffeomorphisms. The Lie algebra $\mathfrak{g}_0$ of the group of exact diffeomorphisms $S_0 \operatorname{Diff}(M)$ is naturally identified (by the map $v \mapsto i_v \mu$) with the space of differential $(n-1)$-forms, which are differentials of $(n-2)$-forms vanishing on ∂M.

Theorem 7.13. *The dual space $\mathfrak{g}_0^*$ of the Lie algebra $\mathfrak{g}_0$ for the group $S_0 \operatorname{Diff}(M)$ is naturally identified with the space of all 2-forms that are differentials of 1-forms on M. This dual space is naturally isomorphic to the quotient $\Omega^1 / \ker(d : \Omega^1 \to \Omega^2)$ of the space of all 1-forms on M, modulo all closed 1-forms. The adjoint and coadjoint representations are the standard diffeomorphism actions on $(n-1)$-forms and on 1-forms.*

Remark 7.14. The *subgroup* generated by the commutators $aba^{-1}b^{-1}$ of a group G is called the *commutant* of G. The commutant of the group $S \operatorname{Diff}(M)$ is the subgroup of the exact diffeomorphisms $S_0 \operatorname{Diff}(M)$ [Ban].

The tangent space to the commutant subgroup of a Lie group is called the *commutant subalgebra* of the Lie algebra. It is generated (as a vector space) by the commutators of the Lie algebra elements. The commutant of the Lie algebra $S \operatorname{Vect}(M)$ of the group $S \operatorname{Diff}(M)$ is the Lie algebra $S_0 \operatorname{Vect}(M)$ [Arn7].

The quotient space $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ of a Lie algebra $\mathfrak{g}$ by its commutant subalgebra $[\mathfrak{g}, \mathfrak{g}]$ is called the one-dimensional homology (with coefficients in numbers) of the Lie algebra $\mathfrak{g}$. Thus the one-dimensional homology of the Lie algebra of divergence-free vector fields on M^n is naturally isomorphic to the De Rham cohomology group

$$H^{n-1}(M^n, \mathbb{R}) = \ker(d : \Omega^{n-1} \to \Omega^n) / \operatorname{Im}(d : \Omega^{n-2} \to \Omega^{n-1})$$

(for a manifold with boundary the $(n-1)$-forms have to vanish on the boundary).

The image of a divergence-free vector field $v \in \mathfrak{g} = S \operatorname{Vect}(M)$ in the one-dimensional homology group

$$\mathfrak{g}/\mathfrak{g}_0 \approx H^{n-1}(M, \mathbb{R}) \approx H_1(M, \mathbb{R})$$

is called the *rotation class* of a vector field.

The rotation class of a divergence-free vector field concentrated along a closed curve γ in M is the homology class of γ (provided that the flux of the field across a transverse section to γ equals 1).

Remark 7.15. In the space $C_1(M)$ of closed curves on a manifold M with boundary there are two interesting subspaces: (i) the curves homologous to zero and (ii) the curves in M homologous to those on ∂M. (Recall that two oriented closed curves a, b on M are homologous if there exists a surface S in M whose boundary is $\partial S = a - b$. Here the minus sign means the reversed orientation.)

One defines two subspaces in the space $\mathfrak{g}$ of all divergence-free vector fields on M tangent to ∂M that correspond to the above-mentioned subclasses of curves: (i) exact fields $v \in \mathfrak{g}_0$ (such that $i_v \mu = d\alpha$, $\alpha|_{\partial M} = 0$) and (ii) semiexact fields $v \in \mathfrak{g}_{se}$, for which $i_v \mu = d\alpha$, $d\alpha|_{\partial M} = 0$.

Theorem 7.16. *The subspaces mentioned above are Lie subalgebras in the Lie algebra $\mathfrak{g}$ of divergence-free vector fields on M tangent to the boundary. Moreover, they are Lie ideals; i.e., the Poisson bracket $\{w, v\}$ of an arbitrary field $w \in \mathfrak{g}$ with a field v from either of the subalgebras $\mathfrak{a}$ ($\mathfrak{a} = \mathfrak{g}_0$ or $\mathfrak{g}_{se}$) belongs to the same subalgebra.*

Proof. A diffeomorphism g from the group $S\,\mathrm{Diff}(M)$ acts on both the field v and the form α in a consistent way, such that the relations $i_v \mu = d\alpha$, $\alpha|_{\partial M} = 0$ and $d\alpha|_{\partial M}$ are preserved under the action. Therefore, every transform Ad_g sends each subalgebra $\mathfrak{a}$ into itself. Let g_t leave the group unity e with velocity $\dot{g}|_{t=0} = w$. The derivative Ad_{g_t} in t takes $\mathfrak{a}$ into itself. This derivative is, up to a sign, the Poisson bracket with the field w. $\qquad\square$

Theorem 7.17. *The dual space $\mathfrak{g}_{se}^*$ (of the Lie algebra $\mathfrak{g}_{se}$ of semiexact divergence-free vector fields) is naturally isomorphic to the quotient space of all 1-forms on M modulo the closed 1-forms on M vanishing on the boundary ∂M.*

7.C. Inertia operator of an n-dimensional fluid

Definition 7.18. A Riemannian metric $(\ ,\ )$ and a measure μ on the compact manifold M (possibly with boundary ∂M) define a nondegenerate *inner product* $\langle\ ,\ \rangle_{\mathfrak{g}}$ on (divergence-free) vector fields v, $w \in \mathfrak{g}$:

$$(7.9) \qquad \langle v, w \rangle_{\mathfrak{g}} := \int_M (v(x), w(x)) \mu.$$

Hence it specifies an invertible *inertia operator* $A : \mathfrak{g} \to \mathfrak{g}^*$ from the Lie algebra $\mathfrak{g}$ to its dual $\mathfrak{g}^*$ such that the image Av of an element $v \in \mathfrak{g}$ is the element of the dual space $\mathfrak{g}^*$ satisfying

$$(7.10) \qquad \langle Av, w \rangle = \langle v, w \rangle_{\mathfrak{g}}$$

for any $w \in \mathfrak{g}$, where $\langle \, , \, \rangle$ on the left-hand side means the pairing between two elements of the dual spaces. (Strictly speaking, nondegeneracy of the inner product implies invertibility of A only on the regular part of $\mathfrak{g}^*$.)

Theorem 7.19. *The inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$ for the Lie algebra $\mathfrak{g} = S\operatorname{Vect}(M)$ of divergence-free vector fields (tangent to the boundary of M) sends a vector field $v \in \mathfrak{g}$ to the coset $[u] \in \mathfrak{g}^*$ containing the 1-form u obtained from the field v by means of the Riemannian "lifting indices": $u(\xi) = (v, \xi)$ for all $\xi \in T_x M$ at any point $x \in M$.*

Proof. The Theorem is proved by comparison of formulas (7.7) and (7.9). In the tangent space $T_x M$ at every point $x \in M$ the Riemannian "lifting indices" of a vector $v(x)$ is exactly the choice of an exterior 1-form on $T_x M$ whose value on any vector $w(x)$ is the Riemannian inner product of $v(x)$ and $w(x)$. $\qquad\square$

For instance, if M is the Euclidean space $\mathbb{R}^n$, the inertia operator sends a vector field $\sum_i v_i(x) \frac{\partial}{\partial x_i}$ to the set of 1-forms $\{\sum_i (v_i(x) + \frac{\partial f}{\partial x_i}) dx_i \mid f \in C^\infty(\mathbb{R}^n)\}$.

Note that the case of a noncompact manifold M, say, $M = \mathbb{R}^n$, needs specification of the decay of the vector fields and differential forms at infinity to make the integrals (7.7) and (7.9) converge.

Definition 7.20. The *energy function* on the Lie algebra $\mathfrak{g}$ of divergence-free vector fields is half the square length of vectors $v \in \mathfrak{g}$ in the inner product $\langle \, , \, \rangle_\mathfrak{g}$:

$$H(v) := \frac{1}{2}\langle v, v \rangle_\mathfrak{g} = \frac{1}{2}\int_M (v, v)\mu = \frac{1}{2}\langle v, Av \rangle.$$

The dual space $\mathfrak{g}^*$ inherits from $\mathfrak{g}$ the nondegenerate inner product $\langle \, , \, \rangle_{\mathfrak{g}^*}$. We define the *energy Hamiltonian* function on $\mathfrak{g}^*$ as half the square length of the elements in $\mathfrak{g}^*$:

$$H([u]) = \frac{1}{2}\langle [u], [u] \rangle_{\mathfrak{g}^*} := \frac{1}{2}\langle A^{-1}[u], [u] \rangle.$$

Here $v = A^{-1}[u]$ is the (divergence-free) vector field related to the coset $[u]$ of 1-forms by means of the Riemannian metric.

Recall that the Euler equations represent the projection of the geodesic flow of the *right*-invariant metric on the group defined by the quadratic form H on the Lie algebra.

Lemma–definition 7.21 [Arn16, OKC]. *The generalized Euler equation on the dual space $\mathfrak{g}^* = \Omega^1/d\Omega^0$ of the Lie algebra of divergence-free vector fields on M has the following form:*

$$(7.11) \qquad\qquad \frac{\partial[u]}{\partial t} = -L_v[u].$$

Here the vector field v is related to the coset $[u]$ of 1-forms by the metric lifting indices on M: $[u] = Av$. Rewritten for a particular representative 1-form $u \in [u]$, the generalized Euler equation becomes

$$(7.12) \qquad \frac{\partial u}{\partial t} = -L_v u - df.$$

The Euler equation on $\mathfrak{g}^*$ is Hamiltonian with respect to the natural Lie–Poisson structure, and minus the energy $-H([u])$ is its Hamiltonian function.

Remark 7.22. One can see that the latter equation on the dual space $\mathfrak{g}^*$ is the image under the inertia operator A of the classical Euler equation (7.1):

$$\frac{\partial v}{\partial t} = -(v, \nabla)v - \nabla p$$

in the Lie algebra $\mathfrak{g}$ of divergence-free vector fields ($\mathrm{div}_\mu\, v = 0$). Here the 1-form u is related by metric lifting indices with the vector field v.

The identification of the equations in the Lie algebra and in its dual is based on the following fact, which we shall prove in Section IV.1.D: The inertia operator A sends the covariant derivative vector field $(v, \nabla)v$ on a Riemannian manifold M to the 1-form $L_v u - \frac{1}{2}d(u(v))$. Then the pressure function p is equal to $f + \frac{1}{2}u(v)$ (modulo an additive constant).

The Helmholtz curl equation $\frac{\partial \omega}{\partial t} = -L_v \omega$ on the space of all exact 2-forms $\omega = du$ on M (see equation (5.3)) is obtained by the exterior differentiation of both sides of equation (7.11). An advantage of the Helmholtz formulation is the pure geometric action on the 2-forms: The form ω is "frozen into the fluid"; i.e., it is transported by the fluid flow exactly, not just modulo some differential (as the 1-form u is).

Corollary 7.23. *If the initial vector field is exact (respectively, semiexact), it will remain exact (respectively, semiexact) for all t.*

This follows from the explicit form of the Euler equation (7.11) and Theorems 7.13 and 7.17 describing the dual spaces to the Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_{se}$.

Proof of Lemma–definition. Equation (7.11) is a Hamiltonian equation on $\mathfrak{g}^*$ with minus the energy $-H([u])$ playing the role of the Hamiltonian function.

Indeed, with respect to the standard linear Lie–Poisson structure, the quadratic Hamiltonian function $-H([u]) = -\frac{1}{2}\langle A^{-1}[u], [u]\rangle$ defines the following equation on $\mathfrak{g}^*$:

$$\frac{\partial [u]}{\partial t} = -\,\mathrm{ad}^*_{A^{-1}[u]}[u];$$

see (6.4). By substituting the explicit form of the inertia operator (Theorem 7.19) and of the coadjoint operator ad^* from (7.6), we complete the proof. $\qquad\square$

Remark 7.24. For an arbitrary Lie group G and an arbitrary (not necessarily quadratic) Hamiltonian functional F on the dual space $\mathfrak{g}^*$ to its Lie algebra $\mathfrak{g}$, the corresponding Hamiltonian equation, with respect to the Lie–Poisson structure, is

$$\dot{m} = \mathrm{ad}^*_{\delta F/\delta m}\, m,$$

where the *variational derivative* $\delta F/\delta m$ of the functional F at the point $m \in \mathfrak{g}^*$ is understood as an element of the Lie algebra $\mathfrak{g}$ and is defined by the relation

$$\frac{d}{d\varepsilon} F(m + \varepsilon w)\Big|_{\varepsilon=0} = \langle w, \delta F/\delta m \rangle,$$

for all $w \in \mathfrak{g}^*$; cf. Example 6.10. For a quadratic functional $F = \frac{1}{2}\langle A^{-1}m, m\rangle$ the variational derivative is $\delta F/\delta m = A^{-1}(m)$.

§8. Proofs of theorems about the Lie algebra of divergence-free fields and its dual

Let M^n be a smooth compact manifold with boundary ∂M and volume element μ. Denote by $\mathfrak{g} = S\,\mathrm{Vect}(M)$ the Lie algebra of all divergence-free vector fields on M that are tangent to ∂M. Let $\Omega^k(M)$ be the space of differential k-forms on M and $\Omega^k(M, \partial M)$ the space of differential k-forms on M whose restriction to ∂M vanishes.

To a vector field v on M we associate the following differential $(n-1)$-form:

$$\omega_v = i_v \mu \in \Omega^{n-1}(M).$$

Lemma 8.1. *The map $v \mapsto w_v$ defines a natural (i.e., invariant with respect to the $S\,\mathrm{Diff}(M)$ action) isomorphism of the vector space of the Lie algebra $\mathfrak{g}$ and the space of all closed differential $(n-1)$-forms on M vanishing on ∂M:*

$$\omega_v \in \ker\left(d : \Omega^{n-1}(M, \partial M) \to \Omega^n(M)\right).$$

Proof. We start with the fundamental *homotopy formula*.

Definition 8.2. The *Lie derivative operator* L_ξ on forms does not change their degree. It evaluates the instantaneous velocity of the form evolved with the medium whose velocity field is ξ. The linear operator L_ξ is expressed in terms of the operators i_ξ and d via the "homotopy formula" $L_\xi = i_\xi \circ d + d \circ i_\xi$.

Now the proof is achieved by applying the homotopy formula to the volume form $\mu \in \Omega^n(M)$. We conclude that

$$L_v \mu = d i_v \mu = d\omega_v;$$

i.e., the flow of v preserves μ if and only if the $(n-1)$-form ω_v is closed. The restriction of ω_v to ∂M is the $(n-1)$-form that gives the flux of the field v across ∂M. The vanishing of ω_v on ∂M is equivalent to the tangency of v to ∂M. $\square$

The statement on duality between the spaces $\mathfrak{g}$ and $\mathfrak{g}^*$ from Sections 3 and 7 has the following precise meaning.

Theorem 8.3 (see also Theorems 3.11, 7.5). *For an n-dimensional compact manifold M with boundary ∂M, the dual space $\mathfrak{g}^*$ (of the Lie algebra $\mathfrak{g}$ of divergence-free vector fields on M tangent to ∂M) is naturally isomorphic to the quotient space $\Omega^1(M)/d\Omega^0(M)$ (of all 1-forms on M modulo full differentials) in the following sense:*

1. *If α is the differential of a function ($\alpha = df$) and $v \in \mathfrak{g}$, then $\iint_M \omega_v \wedge \alpha = 0$.*
2. *If $\iint_M \omega_v \wedge \alpha = 0$ for all $v \in \mathfrak{g}$, then the 1-form α is the differential of a function.*
3. *If $\iint_M \omega_v \wedge \alpha = 0$ for all $\alpha = df$, then $v \in \mathfrak{g}$ (i.e., v is a divergence-free field on M tangent to ∂M).*
4. *The coadjoint action of the group $S\,\mathrm{Diff}(M)$ on the space $\Omega^1(M)/d\Omega^0(M)$ is geometric; i.e., the volume-preserving diffeomorphisms act as changes of coordinates on the (cosets of) 1-forms α.*

Proof. (1) We utilize the Leibniz identity for the exterior derivative d:

$$(8.1) \qquad d(f \wedge \omega_v) = (df) \wedge \omega_v + f(d\omega_v).$$

If $v \in \mathfrak{g}$, then $d\omega_v = 0$ by virtue of Lemma 8.1. Hence

$$\iint_M (df) \wedge \omega_v = \iint_M d(f \wedge \omega_v) = \int_{\partial M} f \wedge \omega_v,$$

according to the Stokes formula. Since $\omega_v|_{\partial M} = 0$, the latter integral equals zero.

(2) Consider a closed curve γ in M (not meeting the boundary ∂M). Let v be a divergence-free vector field that is supported in a narrow solitorus around this curve and whose flux across any transverse to γ is equal to 1. As the thickness ϵ of the solitorus goes to zero we obtain

$$\lim_{\epsilon \to 0} \iint_M \omega_v \wedge \alpha = (-1)^{n-1} \int_\gamma \alpha = 0$$

for an arbitrary closed curve γ. Therefore, α is the differential of a function (namely, of its integral along a curve connecting the current point with a fixed one).

(3) Again, make use of the identity (8.1). Now we know that the integral of $(df) \wedge \omega_v$ over M is equal to zero; hence

$$\iint_M f\,d\omega_v = \int_{\partial M} f \wedge \omega_v$$

for each function f. Pick a δ-type function f supported in a small neighborhood of an interior point of M. Then the integral on the right-hand side vanishes; hence, the left integral is equal to zero as well. It follows that $d\omega_v = 0$ at every interior point of M; that is, the form ω_v is closed (and the field v is divergence free).

Thus both the left and right integrals are zero for an arbitrary function f. In particular, one can take a function whose restriction to ∂M is of δ-type. At every point of the boundary we obtain $\omega_v|_{\partial M} = 0$. In other words,

$$\omega_v \in \ker\left(d : \Omega^{n-1}(M, \partial M) \to \Omega^n(M)\right),$$

and hence $v \in \mathfrak{g}$ by virtue of Lemma 8.1. Item (3) is proved.

(4) The statements (1)–(3) imply that $\mathfrak{g}^* = \Omega^1/d\Omega^0$, since the set of divergence-free vector fields on M with a volume form μ is identified with the space of closed $(n-1)$-forms by means of the correspondence $v \mapsto \omega_v := i_v\mu$. If M has boundary ∂M, then a field v is tangent to ∂M if and only if the form ω_v vanishes on the boundary.

Furthermore, recall that the adjoint action of the group $S\,\mathrm{Diff}(M)$ on a vector field v is a geometric action (change of coordinates) by a diffeomorphism $g \in G$ on v:

$$\mathrm{Ad}_g\, v = g_*v.$$

It follows that the action of the diffeomorphism g on any 1-form α, which is paired with v is also geometric. More precisely, the group coadjoint action on the coset $[\alpha] \in \Omega^1/d\Omega^0 = \mathfrak{g}^*$ representing the 1-form α in the dual space $\mathfrak{g}^*$ is described as follows:

$$\langle v, \mathrm{Ad}_g^*[\alpha]\rangle := \langle \mathrm{Ad}_g\, v, [\alpha]\rangle = \int_M \alpha(g_*v)\mu$$

$$= \int_M (g^*\alpha)(v)\, g^*\mu = \int_M (g^*\alpha)(v)\mu = \langle v, [g^*\alpha]\rangle.$$

Here we make use of the invariance of the volume form: $g^*\mu = \mu$. Thus

$$\mathrm{Ad}_g^*[\alpha] = [g^*\alpha],$$

which completes the proof of the theorem. $\qquad\square$

Consider now the Lie algebra $\mathfrak{g}_0$ of all exact fields v, for which $\omega_v = i_v\mu = d\beta$ for an $(n-2)$-form $\beta \in \Omega^{n-2}(M, \partial M)$ vanishing on the boundary $\beta|_{\partial M} = 0$.

Theorem 8.4. *For an n-dimensional compact manifold M with boundary:*

(1) *If the 1-form α is closed ($d\alpha = 0$), then $\iint_M \omega_v \wedge \alpha = 0$ for all fields $v \in \mathfrak{g}_0$.*

(2) *If $\iint_M \omega_v \wedge \alpha = 0$ for all v in $\mathfrak{g}_0$, then the 1-form α is closed in M.*

(3) *If $\iint_M \omega_v \wedge \alpha = 0$ for all closed 1-forms α on M, then $v \in \mathfrak{g}_0$.*

In other words, the dual space $\mathfrak{g}_0^*$ (of the Lie algebra $\mathfrak{g}_0$ of exact divergence-free vector fields) is naturally isomorphic to the quotient space $\Omega^1(M)/Z^1(M)$ (of all 1-forms on M modulo all closed 1-forms on M).

Proof. (1) Apply the Leibniz identity in the form

$$(8.2) \qquad d(\beta \wedge \alpha) = (d\beta) \wedge \alpha + (-1)^{n-2} \beta \wedge d\alpha,$$

where $\omega_v = d\beta$. If $d\alpha = 0$, then, by virtue of the Stokes formula,

$$\iint_M (d\beta) \wedge \alpha = \int_{\partial M} \beta \wedge \alpha = 0,$$

since $\beta\big|_{\partial M} = 0$.

(2) By making use of (8.2) when $\iint_M (d\beta) \wedge \alpha = 0$, we obtain

$$(-1)^n \iint_M \beta \wedge d\alpha = \int_{\partial M} \beta \wedge \alpha.$$

The latter integral equals zero for every form β vanishing on ∂M. In particular,

$$\iint_M \beta \wedge d\alpha = 0$$

for every $(n-2)$-form β supported compactly inside M. This implies that $d\alpha = 0$.

(3) For a closed form α, we get from (8.2) that

$$0 = \iint_M (d\beta) \wedge \alpha = \iint_{\partial M} \beta \wedge \alpha.$$

Hence, on the closed manifold ∂M the $(n-2)$-form $\beta\big|_{\partial M}$ is orthogonal to every closed 1-form α. By virtue of the Poincaré duality, the $(n-2)$-form $\beta\big|_{\partial M}$ is exact; i.e., there exists an $(n-3)$-form γ on ∂M such that $\beta\big|_{\partial M} = d\gamma$. Extend arbitrarily the $(n-3)$-form γ from ∂M to an $(n-3)$-form $\tilde{\gamma}$ defined on the whole of M (say, extend γ into an ε-neighborhood of ∂M as the pullback $p^*\gamma$, where p is a retraction to the boundary ∂M, and then multiply the result by a cutoff function equal to 1 in the ε-neighborhood and to 0 outside the 2ε-neighborhood). The restriction of $d\tilde{\gamma}$ to ∂M coincides with $\beta\big|_{\partial M}$. Therefore, $\tilde{\beta} = \beta - d\tilde{\gamma}$ is the required $(n-2)$-form on M that vanishes on ∂M and whose differential is $d\beta = i_v \mu$. Thus, $v \in \mathfrak{g}_0$. $\qquad \square$

We leave it to the reader to adjust the above arguments to prove Theorem 7.17.

§9. Conservation laws in higher-dimensional hydrodynamics

The Euler equation of a two-dimensional fluid has an infinite number of conserved quantities (see Section 5). For example, for the standard metric in $\mathbb{R}^2$ one has the

enstrophy invariants

$$J_k(v) = \int_{\mathbb{R}^2} (\operatorname{curl} v)^k \, d^2x = \int_{\mathbb{R}^2} (\Delta \psi)^k \, d^2x, \qquad \text{for} \quad k = 1, 2, \ldots,$$

where ψ is the "stream function" of the vector field v: $v_1 = -\partial \psi / \partial x_2$, $v_2 = \partial \psi / \partial x_1$.

For an ideal fluid filling a three-dimensional simply connected manifold one has the helicity (or Hopf) invariant, which expresses the mutual linking of the trajectories of the vorticity field $\operatorname{curl} v$, and we discuss it in detail in Chapter III. In Euclidean space $\mathbb{R}^3$, it has the form

$$J(v) = \int_{\mathbb{R}^3} (v, \operatorname{curl} v) \, d^3x.$$

This pattern seems to be rather disappointing. One can hardly expect any first integrals in the higher-dimensional case (except for the energy, of course—the kinetic energy is always invariant, being the Hamiltonian function of the Euler equation). It turns out, however, that enstrophy-type integrals do exist for all even-dimensional ideal fluid flows, and so do helicity-type integrals for all odd-dimensional flows. First, we formulate the result for a domain in Euclidean space.

Theorem 9.1 ([Ser1, Dez] **for odd** n, [Tar] **for even** n). *The Euler equation (7.1) of an ideal incompressible fluid on a Riemannian manifold in a bounded domain M in $\mathbb{R}^n$ has*

(1) *the first integral*

$$(9.1a) \qquad \tilde{I}(v) = \int_M \sum_{(i_1 \ldots i_{2m+1})} \varepsilon^{i_1 \ldots i_{2m+1}} v_{i_1} \omega_{i_2 i_3} \cdots \omega_{i_{2m} i_{2m+1}}$$

if the dimension n is odd: $n = 2m + 1$.;

(2) *an infinite number of independent first integrals*

$$(9.1b) \qquad \tilde{I}_k(v) = \int_M \left(\det \|\omega_{ij}\| \right)^k \, d^n x$$

if the dimension n is even: $n = 2m$.

Here v is the velocity vector field of the fluid in M; the functions $\omega_{ij} := \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}$ are components of the vorticity tensor; $\det \|\omega_{ij}\|$ is the determinant of the skew-symmetric matrix $\|\omega_{ij}\|$; the summation in (9.1a) goes over all permutations of the set $(1 \ldots 2m + 1)$; and $\varepsilon^{i_1 \ldots i_{2m+1}}$ is the Kronecker symbol:

$$\varepsilon^{i_1 \ldots i_{2m+1}} = \begin{cases} 1, & \text{if the permutation } (i_1 \ldots i_{2m+1}) \text{ of } (1 \ldots 2m + 1) \text{ is even,} \\ -1, & \text{if the permutation } (i_1 \ldots i_{2m+1}) \text{ of } (1 \ldots 2m + 1) \text{ is odd.} \end{cases}$$

In particular, for $n = 2$ we get from (9.1b)

$$\tilde{I}_k(v) = \int_M \left((\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i})^2 \right)^k \, d^2x = \int_M \left((\Delta \psi)^2 \right)^k \, d^2x = J_{2k}(v),$$

while for $n = 3$ the invariant $\tilde{I}(v)$ from (9.1a) assumes the form of the helicity $J(v)$:

$$\tilde{I}(v) = \int_M \sum_{(i_1 i_2 i_3)} \varepsilon^{i_1 i_2 i_3} v_{i_1} \omega_{i_2 i_3} = J(v).$$

Note that in (9.1b) the parameter k is not necessarily an integer.

This theorem follows, practically without calculations, from the definition of the coadjoint action of the diffeomorphisms group when formulated in the invariant and coordinate-free way.

Define the 1-form u as the inner product with the velocity field v in the sense of the Riemannian metric on a manifold M:

$$u(\xi) = (v, \xi) \qquad \text{for all} \quad \xi \in T_x M.$$

Theorem 9.2 [OKC, KhC]. *The Euler equation (7.1) of an ideal incompressible fluid on a Riemannian manifold M^n (possibly with boundary) with a measure form μ has*

(1) *the first integral*

(9.2a)
$$I(v) = \int_M u \wedge (du)^m$$

in the case of an arbitrary odd-dimensional manifold M $(n = 2m + 1)$; and

(2) *an infinite number of functionally independent first integrals*

(9.2b)
$$I_f(v) = \int_M f\left(\frac{(du)^m}{\mu}\right) \mu$$

in the case of an arbitrary even-dimensional manifold M $(n = 2m)$,

where the 1-form u and the vector field v are related by means of the metric on M, and $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function of one variable.

The fraction $(du)^m/\mu$ for $n = 2m$ is a ratio of two differential forms of the highest degree n. Since the volume form μ vanishes nowhere, the ratio is a well-defined function on M (which may depend on time t). The integral of the function f evaluated at this ratio gives a generalized momentum (i.e., a weighted volume between different level hypersurfaces) of the invariant function $(du)^m/\mu$. The momenta $\tilde{I}_k$ correspond to the choice $f(z) = z^{2k}$. Theorem 9.1 can be obtained from Theorem 9.2 by coordinate rewriting of the differential 2-form du as the matrix $\|\omega_{ij}\|$.

Proof. The trajectories of the Euler equation on $\mathfrak{g}^*$ belong to the coadjoint orbits of the group G. This immediately follows from the Hamiltonian formulation of the Euler equation: The trajectories belong to the symplectic leaves of the Lie–Poisson bracket on $\mathfrak{g}^*$, which are the coadjoint orbits of G.

Now the invariance of the functionals I and I_f along the trajectories follows from

Proposition 9.3. *The following functionals on $\mathfrak{g}^*$ are invariants of the coadjoint action:*

(1) *in case $n = 2m + 1$*

$$I([u]) = \int_M u \wedge (du)^m;$$

(2) *in case $n = 2m$*

$$I_f([u]) = \int_M f\left(\frac{(du)^m}{\mu}\right)\mu,$$

where f is an arbitrary function of one variable, and $[u] \in \Omega^1/d\Omega^0 = \mathfrak{g}^*$ is the coset of a differential 1-form u.

Proof of Proposition. The above functionals are well-defined on $\mathfrak{g}^*$; i.e., they do not depend on the ambiguity in the choice of the representative 1-form u. Indeed, under a change of u to another representative $u + dh$ in the same coset $[u] \in \Omega^1/d\Omega^0$ the form du will not be affected. Hence, the invariants I_f rely merely on the coset $[u]$ of the form u, and so does I, since $I(u + dh) - I(u) = \int dh \wedge (du)^m = 0$.

The coadjoint action of the diffeomorphism group G coincides with the change of variables (see Theorem 7.5) in 1-forms u (or in the corresponding cosets). The integrals I and I_f are defined in a coordinate-free way; hence, they are invariant under the coadjoint action. This completes the proof of Proposition 9.3, as well as of the two preceding theorems. $\square$

Remarks 9.4. (A) At first glance, it seems that one can generate more invariants in odd dimensions by considering shear plane-parallel flows of one dimension higher and using the corresponding even-dimensional invariants. However, the reduction from even to odd dimensions does not provide any new integrals different from (9.2a). The reason is that the invariant (9.2b) for a shear plane-parallel $2m$-dimensional flow obtained from a $(2m - 1)$-dimensional one is trivial: A plane-parallel vector field v induces the 2-form du of rank less than $2m$, since the additional direction lies in the kernel of du. This implies that $(du)^m = 0$, and the corresponding integrals (9.2b) become trivial.

(B) For a noncompact manifold M (say, for the whole space $\mathbb{R}^n$), we should confine ourselves to the class of vector fields and forms decaying fast enough to make convergent the above integrals over M.

The manifold M may be multiconnected. In the case of a non-simply connected manifold M, the cohomology class of the 1-form u (or of the coset $[u]$) corresponding to the vector field v is also invariant (cf. [Arn7]). Other examples of first

integrals of the Euler equation are provided by the number of points or submanifolds in M where the two-form du is degenerate, as well as by the orders of its degeneracy there, and by the invariants of the periodic orbits of the velocity field in the three-dimensional case (periods, Floquet multipliers, etc.).

See also [Ol, Gur] for a discussion of the symmetries, i.e., infinitesimal transformations in the jet spaces, preserving the Euler equations for $n = 2$ and $n = 3$.

A natural by-product of the invariant approach to higher-dimensional hydrodynamics is the following notion of vorticity in n dimensions.

Definition 9.5. The *vorticity form* (or *curl*) of a vector field v on an n-dimensional manifold M is the 2-form $\omega = du$ that is the differential of the 1-form u related to v by means of the chosen Riemannian metric. Depending on the parity of the dimension of M, one can associate to the 2-form ω a vorticity function or a vorticity vector field.

On an even-dimensional manifold M^n ($n = 2m$) the ratio $\lambda = (du)^m / \mu$ is called the *vorticity function* of the field v.

On an odd-dimensional manifold M^n ($n = 2m + 1$), the 2-form $\omega = du$ is always degenerate, and the *vorticity vector field* is the kernel vector field ξ of the vorticity form ω: $i_\xi \mu = \omega^m$.

Example 9.6. In Euclidean space $\mathbb{R}^{2m}$ with standard volume form, the vorticity function of a vector field v is

$$\lambda = \sqrt{\det \|\omega_{ij}\|},$$

and for $n = 2m = 2$ it is the standard definition of the vorticity function

$$\operatorname{curl} v = \partial v_1 / \partial x_2 - \partial v_2 / \partial x_1.$$

In $\mathbb{R}^{2m+1}$ with the Euclidean metric the vorticity field ξ has the coordinates

$$\xi_j = \sum_{(j\, i_1 \ldots i_{2m})} \varepsilon^{j\, i_1 \ldots i_{2m}} \omega_{i_1 i_2} \cdots \omega_{i_{2m-1} i_{2m}},$$

where $\varepsilon^{j\, i_1 \ldots i_{2m}}$ is the Kronecker symbol, and the summation is over all permutations of $(1 \ldots 2m+1)$. In $\mathbb{R}^3$ this expression gives the classical definition of the vorticity field $\xi = \operatorname{curl} v$.

Proposition 9.7. *The vorticity vector field ξ and the vorticity function λ are transported by the Euler flow on, respectively, odd- or even-dimensional manifolds.*

Proof. Indeed, the coadjoint action is geometric, and it changes coordinates in the 2-form du. Thus du is transported by the flow, while the volume form μ is invariant under it. Hence, the vorticity vector field and function, defined in terms of these two objects, are transported by the incompressible flow as well. □

The above statement is based on the Helmholtz evolution equation valid for the 2-form $\omega = du$: $\frac{\partial \omega}{\partial t} = -L_v \omega$. It means that the substantial derivative of ω vanishes, or that this 2-form is transported by the flow.

Remark 9.8. The above integrals are invariants of the coadjoint representation of the corresponding Lie groups (the so-called *Casimir functions*); i.e., they are invariants of the Hamiltonian equations with respect to the Lie–Poisson structure on $\mathfrak{g}^*$ for an arbitrary choice of the Hamiltonian function. The integrals $I([u])$ and $I_f([u])$ do not form a complete set of continuous invariants of coadjoint orbits. One can construct parametrized families of orbits with equal values of these functionals, similar to those described in Section 5 for $n = 2$ and in Chapter III for $n = 3$. For instance, in odd dimensions the flow preserves not only the integral (9.2a) over the entire manifold M, but also the integrals

$$I_C(v) = \int_C u \wedge (du)^m$$

over every invariant set C of the vorticity vector field for the instantaneous velocity v. This follows immediately from the Stokes formula, applied to such an invariant set, and from the observation that the restriction of $(du)^m$ to the boundary of any invariant set vanishes.

A precise description of the coadjoint orbits for the diffeomorphism groups still remains an unsolved and intriguing problem. In particular, one may think that the closure of a coadjoint orbit for $n = 3$ could contain an open part of a level set of the integral $I(v)$ in some topology. Physically, this would mean that in the three-dimensional case preservation of vorticity is not as restrictive on the particles' permutations realized by the flow as it is in the two-dimensional case.

The reason for this conjecture is the following result on *local invariants* of the coadjoint orbits (i.e., the local description of isovorticed fields) in ideal hydrodynamics.

Theorem 9.9.

(1) *The vorticity function λ is the only local invariant of the coadjoint orbits of the group of volume-preserving diffeomorphisms of an even-dimensional manifold M^n ($n = 2m$) at a generic point.*

(2) *For odd-dimensional M^n ($n = 2m + 1$) there are no local invariants of the coadjoint orbits of the group $S\,\mathrm{Diff}\,(M^n)$; i.e., at a generic point of the manifold the cosets belonging to different coadjoint orbits can be identified by means of a volume-preserving diffeomorphism.*

For instance, in the two-dimensional case $n = 2$, the set of isovorticed fields is fully described by their vorticity function $\mathrm{curl}\, v = \partial v_1 / \partial x_2 - \partial v_2 / \partial x_1$. In the three-dimensional case $n = 3$, the vorticity vector field can be rectified in the vicinity of every nonzero point by a volume-preserving change of coordinates,

and hence has no local invariants. The coadjoint invariant for the odd-dimensional case, provided by Proposition 9.3(2) has genuine global nature: It expresses the linking of vortex trajectories in the manifold; see Chapter 3.

Proof. In the coordinate-free language, the local invariants of the coadjoint action are associated to a coset $[u] \in \mathfrak{g}^*$ of 1-forms in a small neighborhood on a manifold equipped with a volume form. The local invariants of the coset $[u] = \{u + df\}$ (i.e., a 1-form u up to addition of the function differential) are the same as the local invariants of the 2-form du, since taking the differential of the 1-form kills the ambiguity df.

Therefore, the problem reduces to the description of invariants of a closed 2-form in the presence of a volume form μ, i.e., the form of degree n. If n is even, one can think of du as a symplectic form. The pair (du, μ) has the following invariant function associated to them: the symplectic volume $\lambda = (du)^m/\mu$, where $n = 2m$. This volume is nothing but the vorticity function. The uniqueness of this invariant in a generic point immediately follows from the Darboux theorem: By a (non-volume-preserving) change of variables the 2-form du transforms to $\sum_i dp_i \wedge dq_i$, while the volume form becomes $d^m p \, d^m q/\lambda(p, q)$; see [A-G].

If n is odd, in a generic point there is no invariant for the pair (du, μ). Indeed, a nondegenerate 2-form du again transforms to $du = \sum_{i=1}^m dp_i \wedge dq_i$ in $\mathbb{R}^{2m+1} = \{(p, q, z)\}$, according to a version of the Darboux theorem. Then by changing the coordinate $z \to z' = h(p, q)$ one can reduce the volume form to $\mu = d^m p \wedge d^m q \wedge dz$ without further changing du. Thus a generic pair (du, μ) has a unique canonical form.   □

The above theorem does not imply that other invariants that are *integrals of local densities* over the flow domain could not exist. We conjecture that there are no new integral invariants either for the Euler equation or for the coadjoint orbits of the diffeomorphism groups. The *integral invariants* of a vector field v are functionals of the form $\int_M f(v) \, d^n x$. The density function f is called *local* if it depends on only a finite number of partial derivatives of v. D. Serre showed in [Ser3] that for the 3D Euler equation all integral invariants whose densities depend on velocity and its first partial derivatives are indeed combinations of helicity and energy.

Remark 9.10. The Casimir functions, i.e., invariants of the coadjoint representation, allow one to study the nonlinear stability problems by Routh-type methods (see Chapter II). The information about the orbits can be helpful in the study of the Cauchy problem in high-dimensional hydrodynamics. The different number of invariants in odd and even dimensions apparently indicates that the existence theorem in odd- and even-dimensional hydrodynamics should require essentially different arguments.

Instead of writing the Euler equation as an evolution of a coset $[u]$, one can choose the special 1-form $\bar{u}$ for which the action of the flow is geometric. This would allow one to write the invariants in odd dimensions in the same way as for $n = 2m$, since, say, the ratio $\bar{u} \wedge (d\bar{u})^{2m}/\mu$ is transported by the flow for such a choice of $\bar{u}$. To find the corresponding evolution of $\bar{u}$ one has actually to solve the

Euler equation for the velocity field [Ose2, GmF]. Such invariants are similar to Lagrangian coordinates of fluid particles.

Note that the existence of an infinite series of integrals for a flow of an ideal even-dimensional fluid does not imply complete integrability of the corresponding hydrodynamic equations. These invariants merely specify the coadjoint orbits (generally speaking, infinite-dimensional) where the dynamics takes place. For the evolution on the orbit itself we know just the energy integral, while integrability requires specification of an infinite number of integrals.

On the other hand, the Euler hydrodynamic equations in the plane admit finite-dimensional truncations of arbitrarily large size that turn out to be integrable Hamiltonian systems [MuR]. We discuss finite-dimensional approximations of classical hydrodynamic equations in Section 11. In Section VI.3 we will show how knot theory can be regarded as a part of coadjoint orbit classification for the group $S \operatorname{Diff}(M^3)$. Knots correspond to highly degenerate orbits of differential 2-forms supported on curves in a three-dimensional manifold M. Knot invariants with respect to isotopies become Casimir invariants for such degenerate orbits.

§10. The group setting of ideal magnetohydrodynamics

Magnetic fields in perfectly conducting plasma or magma are among the main objects of study in astrophysics and geophysics. In the idealized setting, an inviscid incompressible fluid obeying hydrodynamical principles transports a magnetic field. In turn, the medium itself experiences a reciprocal influence of the magnetic field. The evolution is described by the corresponding system of Maxwell's equations.

10.A. Equations of magnetohydrodynamics and the Kirchhoff equations

Definition 10.1. We assume first that an electrically conducting fluid fills some domain M of the Euclidean three-dimensional space $\mathbb{R}^3$. The fluid is supposed to be incompressible with respect to the standard volume form $\mu = d^3x$, and it transports a divergence-free magnetic field **B**. Then, the evolution of the field **B** and of the fluid velocity field v is described by the system of ideal *magnetohydrodynamics (MHD) equations*

$$(10.1) \quad \begin{cases} \frac{\partial v}{\partial t} = -(v, \nabla)v + (\operatorname{curl} \mathbf{B}) \times \mathbf{B} - \nabla p, \\ \frac{\partial \mathbf{B}}{\partial t} = -\{v, \mathbf{B}\}, \\ \operatorname{div} \mathbf{B} = \operatorname{div} v = 0. \end{cases}$$

Here the second equation is the definition of the "frozenness" of the magnetic field **B** into the medium, and $\{\,,\,\}$ denotes the Poisson bracket of two vector fields.

In the first equation the pressure term ∇p is uniquely defined by the condition div $\partial v/\partial t = 0$, just as it is for the Euler equation in ideal hydrodynamics. The term (curl $\mathbf{B}$) × $\mathbf{B}$ represents the Lorentz force. On a unit charge moving with velocity $\mathbf{j}$ in the magnetic field $\mathbf{B}$ there acts the Lorentz force $\mathbf{j} \times \mathbf{B}$. On the other hand, the electrical current field $\mathbf{j}$ is equal to curl $\mathbf{B}/4\pi$ according to Maxwell's equation [Max]. The coefficients in equation (10.1) are normalized by a suitable choice of units.

The *total energy* E of the MHD system is the sum of the kinetic and magnetic energy:

$$(10.2) \qquad E := \frac{1}{2}\langle v, v \rangle + \frac{1}{2}\langle \mathbf{B}, \mathbf{B} \rangle.$$

Remark 10.2. One can view the *Kirchhoff equations* [Kir]

$$(10.3) \qquad \begin{cases} \dot{p} = p \times \omega, \\ \dot{m} = m \times \omega + p \times u \end{cases}$$

for a rigid body moving in a fluid as a finite-dimensional analogue of the magnetohydrodynamics (just as the classical rigid body with a fixed point is analogous to the ideal fluid dynamics); see [V-D, DKN]. The fluid is ideal, incompressible, and at rest at infinity, and the fluid motion itself is supposed to be potential. The energy of a body in a fluid is

$$(10.4) \qquad H = \frac{1}{2}\left(\sum a_i m_i^2 + \sum b_{ij}(p_i m_j + m_i p_j) + \sum c_{ij} p_i p_j\right).$$

The variables m and p are the total *angular momentum* and the *vector momentum* of the body–fluid system in a moving coordinate system rigidly attached to the body; $u^i = \partial H/\partial p_i$, $\omega^i = \partial H/\partial m_i$. The energy is quadratic in m, p; it is assumed to be positive; and it defines a Riemannian metric on the group $E(3)$ of all motions in three-dimensional Euclidean space.

In the case of magnetohydrodynamics the total energy is to be considered as the Riemannian metric on the configuration space, which is the *semidirect product* of the diffeomorphism group $S\,\mathrm{Diff}(M)$ and the dual space $\mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$. This space and metric are defined below.

10.B. Magnetic extension of any Lie group

Consider the following example: the one-dimensional Lie group G of all dilations of a real line $x \mapsto bx$. The composition of two dilations with factors b_1 and b_2 defines the dilation with the factor $b_1 b_2$. We will call the two-dimensional group F of all affine transformations of the line $x \mapsto a + bx$ the *magnetic extension* of the group G. Now, the composition of two affine transformations $x \mapsto a_1 + b_1 x$ and $x \mapsto a_2 + b_2 x$ sends every point x to the point

$$a_2 + b_2(a_1 + b_1 x) = (a_2 + a_1 b_2) + (b_1 b_2)x.$$

Hence the group multiplications of the pairs (a, b) that constitute the magnetic extension group F, is

$$(a_2, b_2) \circ (a_1, b_1) = (a_2 + a_1 b_2, b_1 b_2);$$

see also Section IV.1.A. The general description of magnetic extensions below can be regarded as a group formalization of this construction.

Let G be an arbitrary Lie group. We associate to this group a new one, called the *magnetic extension* of the group G, in the following way. The elements of the new group are naturally identified with all points of the phase space T^*G whose configuration space is G.

The group G acts naturally on itself by left translations, as well as by right ones. The left and right shifts commute with each other. Hence, right-invariant vector (or covector) fields are taken to right-invariant ones under left translations, while left-invariant fields are sent to left-invariant ones by right translations.

Extend every covector on G, i.e., an element α_g of the cotangent bundle T^*G at $g \in G$, to the right-invariant section (covector field) α on the group. Define the action of this covector α_g on the phase space T^*G as follows. First add to every covector in T^*G at h the value of the right-invariant section α at h. Then apply the left shift of the entire phase space T^*G by g (Fig. 8).
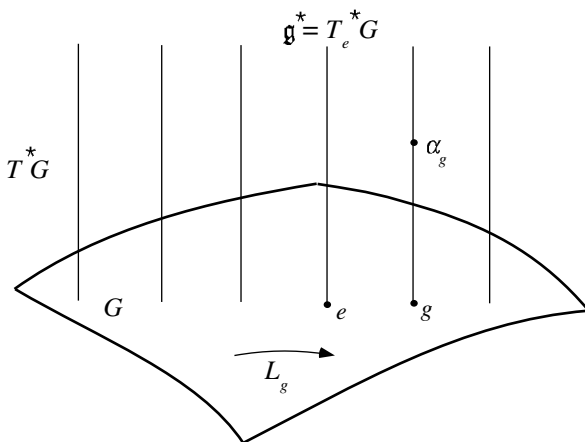
FIGURE 8.   Cotangent bundle T^*G turns into a Lie group.

Theorem 10.3. *The result of two consecutive applications of two cotangent vectors of the group coincides with the action of a new cotangent vector. This composition makes the space T^*G into a Lie group.*

Proof. The composition of two left shifts on a group is a left shift as well. The operator T_2 of addition of the second right-invariant covector field after the first

left shift L_1 coincides with the addition T_1 of another right-invariant covector field preceding the first left shift. Namely, the new covector field is the image of the second covector field under the action of the inverse L_1^{-1} of the first left shift L_1:

$$L_2 T_2 L_1 T_1 = L_2 L_1 \tilde{T}_2 T_1,$$

where $\tilde{T}_2 = L_1^{-1} T_2 L_1$. The sum of this new field with the first right-invariant field is the right-invariant covector field that is to be added to each covector of T^*G before the left translation–composition to obtain the result of the two consecutive applications of two cotangent vectors.   □

Note that a right-invariant field is determined by its value at the group identity. Hence, the phase space T^*G is diffeomorphic to the direct product $G \times \mathfrak{g}^*$ of the group G and of the dual space $\mathfrak{g}^*$ to its Lie algebra (the cotangent bundle of any Lie group is naturally trivialized). However, the *group T^*G constructed above is not the direct product* of the group G and the commutative group $\mathfrak{g}^*$.

Consider a group element $(\psi, b) \in T^*G$, i.e., the composition of the addition of a right-invariant field whose value at the group identity is some covector b followed by the left shift by ψ, and similarly, another element $(\phi, a) \in T^*G$.

Theorem 10.3′. *The composition of (ψ, b) followed by (ϕ, a) is the left shift by $\phi \circ \psi$ preceded by adding the right-invariant field whose value at the identity is the covector $\mathrm{Ad}_\psi^* a + b$.*

Proof. The left translation by ψ^{-1} of the right-invariant field generated by a at the identity is the right-invariant field whose value at the identity is

$$L_\psi^* R_{\psi^{-1}}^* a = (R_{\psi^{-1}} L_\psi)^* a = (\mathrm{Ad}_\psi)^* a.$$

□

Definition 10.4. The *magnetic extension $F = G \ltimes \mathfrak{g}^*$* of a group G is the group of pairs $\{(\phi, a) \mid \phi \in G, a \in \mathfrak{g}^*\}$ with the following group multiplication between the pairs:

$$(10.5) \qquad (\phi, a) \circ (\psi, b) = (\phi \circ \psi, \mathrm{Ad}_\psi^* a + b).$$

The definition of the Lie algebra corresponding to the magnetic extension F follows immediately.

Definition 10.5. The Lie algebra $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{g}^*$, corresponding to the (magnetic extension) group $F = G \ltimes \mathfrak{g}^*$, is the vector space of pairs $(v \in \mathfrak{g}, a \in \mathfrak{g}^*)$ endowed with the following Lie bracket:

$$(10.6) \qquad [(v, a), (w, b)] = ([v, w], \mathrm{ad}_w^* a - \mathrm{ad}_v^* b),$$

where $[v, w]$ is the commutator of the elements v and w in the Lie algebra $\mathfrak{g}$ itself, and $\mathrm{ad}_w^* a$ is the coadjoint action of the algebra $\mathfrak{g}$ on its dual space $\mathfrak{g}^*$.

The magnetic extension is a particular case of the notion of a *semidirect product* of a Lie group G, or a Lie algebra $\mathfrak{g}$, by a vector space V where this group or algebra acts. In the general situation the operators Ad and ad of the coadjoint action in (10.5–6) are to be replaced by the action of the corresponding group or algebra elements on the vector space V; see Section VI.2 and [MRW].

Example 10.6. The symmetry group $E(3)$ of a rigid body in a fluid is the magnetic extension $E(3) = SO(3) \ltimes \mathbb{R}^3$ of the group $SO(3)$ of all rotations of the three-dimensional space by the dual space $\mathfrak{so}(3) = \mathbb{R}^3$. As we shall see in the next section, the Kirchhoff equations (10.3) describe the geodesics on this group $E(3)$ with respect to the *left*-invariant metric defined by the energy E from (10.2).

Example 10.7. The configuration space of magnetohydrodynamics is the magnetic extension $F = S\,\mathrm{Diff}(M) \ltimes (\Omega^1/d\Omega^0)$ of the group $G = S\,\mathrm{Diff}(M)$ of volume-preserving diffeomorphisms of a manifold M and of the corresponding dual space $\mathfrak{g}^* = \Omega^1/d\Omega^0$. The group coadjoint action $\mathrm{Ad}^*_{\psi}\, a = \psi_* a$ in (10.5) is the change of coordinates by the diffeomorphism ψ in the coset a of 1-forms. The corresponding operator ad in (10.6) of the coadjoint action of the Lie algebra is the Lie derivative operator on the cosets: $\mathrm{ad}^*_w\, a = L_w a$.

The MHD equations (10.1) are the geodesic equations on the group F with respect to the *right*-invariant metric defined by the magnetic energy E (10.4); see Theorem 10.9 below.

Remark 10.8. The above definitions can be applied to any manifold M (of arbitrary dimension) equipped with a volume form. Correspondingly, one can define the equations of magnetohydrodynamics on the manifold once one specifies a Riemannian metric on M whose volume element is the given volume form. The only operation that has not yet been specified in the general setting is the cross product, and this can be done using the isomorphism $*$ of k- and $(n-k)$-polyvector fields induced by the metric on a manifold M of any dimension n [DFN]. We refer to [M-W, KhC] for generalizations of the MHD formalism to other dimensions.

Notice also that in the two-dimensional case one has two options for generalizations of equations (10.1): The magnetic field **B** can be regarded as a divergence-free vector field, or, alternatively, as a closed two-form on M. The latter is the same as a function on the two-dimensional manifold M. According to these two possibilities, one has two different systems of equations (see, e.g., the Hamiltonian formulations of MHD presented in [MoG, H-K, ZeK, Ze2]).

10.C. Hamiltonian formulation of the Kirchhoff and magnetohydrodynamics equations

Theorem 10.9 [V-D, MRW]**.**

(1) *The equations of the magnetic hydrodynamics (10.1) are Hamiltonian equations on the space $\mathfrak{f}^*$ dual to the Lie algebra $\mathfrak{f} = S\,\mathrm{Vect}(M) \ltimes (\Omega^1/d\Omega^0)$*

(2) *The Kirchhoff equations (10.3) are Hamiltonian equations on the space*
$\mathfrak{e}(3)^*$ *dual to the Lie algebra* $\mathfrak{e}(3) = \mathfrak{so}(3) \ltimes \mathbb{R}^3$

relative to the standard Lie–Poisson bracket. The Hamiltonian functions are the quadratic forms on the dual spaces defined by the total energy $-E$ *or* H *(formulas (10.2) and (10.4), respectively).*

Proof. Consider the MHD system in three dimensions. The dual Lie algebra $\mathfrak{f}^*$ of the magnetic extension F as a set of pairs is $\mathfrak{f}^* = \{([u], \mathbf{B}) \mid [u] \in \mathfrak{g}^* = \Omega^1/d\Omega^0, \mathbf{B} \in \mathfrak{g} = S\,\mathrm{Vect}(M)\}$. The explicit formula for the coadjoint action of the Lie algebra $\mathfrak{f}$ on its dual space $\mathfrak{f}^*$ is

$$(10.7) \qquad \mathrm{ad}^*_{(v,[\alpha])}([u], \mathbf{B}) = (L_v[u] - L_{\mathbf{B}}[\alpha], -L_v\mathbf{B}).$$

Here $-L_v\mathbf{B} = \{v, \mathbf{B}\}$ is the Poisson bracket of the two vector fields.

The Riemannian metric on M defines the isomorphism $A : \mathfrak{g} \to \mathfrak{g}^*$ between the space $\mathfrak{g} = S\,\mathrm{Vect}(M)$ of divergence-free vector fields and the dual space $\mathfrak{g}^* = \Omega^1/d\Omega^0$; see Section 7. It induces the inner product on the magnetic extension algebra $\mathfrak{f}$, as well as on its dual space $\mathfrak{f}^*$. The corresponding quadratic form of the energy E on $\mathfrak{f}^*$ is

$$E([u], \mathbf{B}) = \frac{1}{2}\langle [u], A^{-1}[u]\rangle + \frac{1}{2}\langle \mathbf{B}, A(\mathbf{B})\rangle.$$

Thus the Euler equation on $\mathfrak{f}^*$ with the Hamiltonian function $-E$, i.e., the geodesic equation for the corresponding right-invariant metric on the group F, is

$$(10.8) \qquad \begin{cases} \frac{\partial [u]}{\partial t} = -L_v[u] + L_{\mathbf{B}}[b], \\ \frac{\partial \mathbf{B}}{\partial t} = -\{v, \mathbf{B}\}, \end{cases}$$

where the vector field v and the coset $[b]$ are, respectively, related to the coset $[u]$ and the magnetic vector field $\mathbf{B}$ by means of the inertia operator:

$$v = A^{-1}[u], \qquad [b] = A(\mathbf{B}).$$

Equations (10.1) are equivalent to their intrinsic form (10.8), as the following statement shows.

Lemma 10.10. *The operator A^{-1} of the Riemannian identification of divergence-free vector fields and the cosets of 1-forms on the manifold M takes the coset $L_{\mathbf{B}}[b]$ (i.e., $L_{\mathbf{B}}A(\mathbf{B})$) to the field* $\mathrm{curl}\,\mathbf{B} \times \mathbf{B}$*, provided that the volume form μ is defined by the Riemannian volume element on M.*

The verification of the latter in a local coordinate system is straightforward.

$\square$

Corollary 10.11. *The inner product*

$$J(v, \mathbf{B}) = \int_M (v, \mathbf{B})\mu$$

of the fluid velocity and the evolved magnetic field is the first integral of the motion defined by the MHD equation (10.1).

A way to prove this statement is to differentiate the quantity J along the vector field given by (10.1). It is, however, a consequence of the following, more general, observation.

Corollary 10.11′ [V-D]**.** *Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. Then the quadratic form*

$$J(u, \alpha) = \langle u, \alpha \rangle,$$

on the dual space $\mathfrak{f}^* = \{(u, \zeta) \mid u \in \mathfrak{g}^*, \zeta \in \mathfrak{g}\}$ of the magnetic Lie algebra $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{g}^*$ is an invariant of the coadjoint representation of the Lie group $F = G \ltimes \mathfrak{g}^*$. Here $\langle u, \alpha \rangle$ stands for the pairing of two elements of the dual spaces $\mathfrak{g}$ and $\mathfrak{g}^*$.

Proof. The invariance of the quadratic form J is verified by direct calculation using the operators of coadjoint action of the group F.

Applying it to the MHD group $F = S\,\mathrm{Diff}(M) \ltimes \Omega^1/d\Omega^0$, we prove the invariance of $J(v, \mathbf{B})$ on the coadjoint orbits of F. The conservation of $J(v, \mathbf{B})$ on the trajectories of (10.1) follows from the Hamiltonian formulation of the equations: The trajectories are tangent to the coadjoint orbits of the group F.

$\square$

Remark 10.12. The quadratic form $J(v, \mathbf{B}) = \int_M u(\mathbf{B})\mu$ (called *cross-helicity*) has a simple topological meaning, being the asymptotic linking number of the trajectories of the magnetic field $\mathbf{B}$ with the trajectories of the vorticity field curl v. It is similar to the total *helicity* of an ideal fluid (which measures the asymptotic mutual linking of the trajectories of the fluid vorticity field) or the magnetic helicity (measuring the linking of magnetic lines); see Chapter III.

The vorticity field of an ideal incompressible fluid is transported (convected) by the fluid flow, and topological invariants of the field are preserved in time. However, unlike the helicity in hydrodynamics, the conservation of the mutual linking between the magnetic and vorticity fields in MHD flow is somewhat unexpected, since curl v in magnetohydrodynamics is not frozen (in contrast to the magnetic field $\mathbf{B}$). The evolution changes the field v (and hence curl v as well) by some additive summand, which depends on $\mathbf{B}$, but it turns out that the mutual linking of the vorticity field curl v and the magnetic field $\mathbf{B}$ is preserved (see [VlM] for more detail).

It would be of special interest to find a description of Casimir functions for magnetohydrodynamics. In particular, one wonders whether there exists an MHD analogue of the complete classification of local invariants of the coadjoint action for ideal hydrodynamics (Theorem 9.9) and what are the integral invariants defined by local densities.

§11. Finite-dimensional approximations of the Euler equation

The effort to give a comprehensive finite-dimensional picture of hydrodynamical processes has a long history: Any attempt to model the Euler equation numerically leads to some kind of truncation of the continuous structure of the equation in favor of a discrete analogue.

According to the main line of this book, we will concentrate on the methods preserving the Hamiltonian structure of the Euler equation and will leave aside numerous (and equally fruitful) methods related to difference schemes or to series expansions of the solutions. We discuss the Galerkin-type approximations for solutions of the Navier–Stokes equation in the next section.

11.A. Approximations by vortex systems in the plane

For numerical purposes one usually starts with the Euler equation written in the Helmholtz form

$$(11.1) \qquad \dot{\mathbf{w}} = -\{\mathbf{v}, \mathbf{w}\},$$

which describes the evolution of the vorticity field $\mathbf{w} = \operatorname{curl} \mathbf{v}$ frozen into a flow with velocity $\mathbf{v}$, and in which $\{\mathbf{v}, \mathbf{w}\}$ stands for the Poisson bracket of two divergence-free vector fields $\mathbf{v}$ and $\mathbf{w}$.

For a two-dimensional incompressible flow in a domain of $D \subset \mathbb{R}^2$, the right-hand side of the equation is the Poisson bracket of the *stream function* ψ and *vorticity function* $\omega = \Delta \psi$ of the vector field $\mathbf{v} = \operatorname{sgrad} \psi$:

$$(11.2) \qquad \dot{\omega} = -\{\psi, \omega\},$$

where $v_x = -\partial \psi / \partial y$, $v_y = \partial \psi / \partial x$, and $\{\psi, \omega\} = \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x}$. Indeed, $-\{\mathbf{v}, \mathbf{w}\} = L_{\mathbf{v}} \mathbf{w}$ for any vector field $\mathbf{w}$. For a plane velocity field $\mathbf{v} = (v_1, v_2, 0)$, the vorticity is a function (the third component of the vorticity field $\mathbf{w} = (0, 0, \omega)$), and $L_{\mathbf{v}} \mathbf{w} = (0, 0, -L_{\mathbf{v}} \omega)$ (vector fields are transported forward, but functions are transported backward). Finally, we transform equation (11.1) to equation (11.2) by applying the definition of the Hamiltonian (or stream) function ψ for the field $\mathbf{v}$: $-L_{\mathbf{v}} \omega = -\{\psi, \omega\}$.

The first approximation scheme we discuss for this equation goes back to Helmholtz. It replaces a smooth vorticity function ω in $D \subset \mathbb{R}^2$ by a collection of *vortices*, i.e., by a vorticity distribution supported at a finite number of points in D. Note that this scheme is also applied to a vorticity field in $\mathbb{R}^3$, where a smooth field is approximated by a singular one supported on a finite number of straight lines.

As we shall see, the motion of such isolated vortices is governed by a Hamiltonian system of ordinary differential equations. The corresponding Hamiltonian function is the logarithmic potential, i.e., a linear combination of the logarithms of the distances between the vortices (coefficients being the products of the vortex intensities).

Consider N vortices with *circulations* (i.e., the velocity circulation around the vortex) k_i, $i = 1, \ldots, N$, in the plane $\mathbb{R}^2$. Then the vorticity at any moment will be concentrated at N points, and the circulations at each of them will remain constant forever. Denote the (Cartesian) coordinates of the vortices in the plane by $z_i := (x_i, y_i), i = 1, \ldots, N$. It is convenient to write down the evolution of vortices as a dynamical system in the configuration space for the N-vortex system, the space $\mathbb{R}^{2N}$ with coordinates $(x_1, y_1, \ldots, x_N, y_N)$ and symplectic structure $\sum k_i \, dy_i \wedge dx_i$.

Proposition 11.1 (see, e.g., [Kir]). *The vortex evolution is then given by the following system of Hamiltonian canonical equations:*

(11.3)
$$
\begin{cases}
k_i \, \dot{x}_i = \frac{\partial H}{\partial y_i}, \\
k_i \, \dot{y}_i = -\frac{\partial H}{\partial x_i},
\end{cases}
$$

$1 \le i \le N$, *where the Hamiltonian function H is*

$$
H = -\frac{1}{\pi} \sum_{i < j} k_i k_j \ln |z_i - z_j|,
$$

and $|z - z_i| = \sqrt{(x - x_i)^2 + (y - y_i)^2}$.

Proof. On the plane, the vorticity function ω describing a point vortex system has the form of a linear combination of the δ-functions:

$$
\omega(z) = \sum_{i=1}^{N} k_i \, \delta(z - z_i).
$$

To derive the equation of vortex evolution we first find the corresponding stream function ψ such that $\Delta \psi = \omega$. Our choice of the vorticity ω implies that the stream function is the linear combination of the fundamental solutions of the two-dimensional Laplace equation

$$
\psi(z) = \frac{1}{2\pi} \sum_{i=1}^{N} k_i \ln |z - z_i|
$$

(plus any harmonic function, which is assumed to be zero due to the vanishing boundary conditions at infinity of $\mathbb{R}^2$).

By substituting these explicit expressions for $\omega(z)$ and $\psi(z)$ into the Euler equation (11.2), we obtain that every vortex will evolve according to the following law:

$$
k_j \dot{z}_j = \text{sgrad} \big|_{z=z_j} \psi(z) = \frac{1}{2\pi} \sum_{i=1, i \neq j}^{N} k_i \, \text{sgrad} \big|_{z=z_j} (\ln |z - z_i|).
$$

The function $\psi(z)$ has a singularity at z_j, but this does not affect the motion of this vortex. This is why we can subtract the contribution of the vortex influence on itself when writing its evolution equation.

Rewritten in (x_i, y_i)-coordinates, the latter gives us the required Hamiltonian system (11.3).   □

According to Helmholtz [Helm], in the case of $N = 2$, the two vortices rotate uniformly in the plane $\mathbb{R}^2$ about their common "mass center" (or rather "center of vorticity") $z = (k_1 z_1 + k_2 z_2)/(k_1 + k_2)$. In particular, if the circulations k_1 and k_2 are of the same sign, then the "mass center" is situated between the vortices, while if they are of opposite signs, then the "mass center" lies on the continuation of the line joining the vortices. If $k_1 = -k_2$, then the point vortices travel with equal velocity in parallel directions perpendicular to the line joining them.

The three-vortex problem ($N = 3$) also turns out to be integrable (unlike the classical three-body problem of gravitating mass points; see, e.g., [Poi1, Poi3]). This has already been pointed out by Kirchhoff [Kir] and illuminated in the dissertation of Gröbli [Grö], where one can find equations for evolution of the sides of the vortex triangle and explicit formulas for several special cases. An elaborate treatment of the history of the problem of three vortices can be found in [ART]. The motion of three point vortices on a sphere is considered in [KiN]. See also [Brd, BFS] for the statistical mechanics approach and [NewP] for the application of the Hannay–Berry phase (Section IV.1) to this problem.

11.B. Nonintegrability of four or more point vortices

For a general N, the Hamiltonian equations of motion (11.3) have the following four first integrals:

$$I_1 = H, \ I_2 = \sum_{i=1}^{N} k_i x_i, \ I_3 = \sum_{i=1}^{N} k_i y_i, \quad I_4 = \sum_{i=1}^{N} k_i (x_i^2 + y_i^2).$$

However, these integrals are not in involution; that is, their Poisson brackets are not zero, and the system with four vortices is, generally speaking, nonintegrable [Zig1]. More precisely, the following statement holds.

Let M^5 be the (five-dimensional) manifold of all nonsingular configurations of four vortices (i.e., $z_i \neq z_j$ if $i \neq j$). This manifold is the quotient of all ordered quadruples of points in $\mathbb{R}^2$ over the three-dimensional group $E(2)$ of all motions of the plane. The quotient M^5 is a smooth manifold, since the group $E(2)$ acts on the set of ordered quadruples without fixed points.

Theorem 11.2 [Zig1]. *For sufficiently small $\epsilon > 0$, the dynamical system of four vortices with circulations $|k_i - 1| < \epsilon$, $i = 1, 2, 3$, $|k_4| < \epsilon$, has no analytic first integral in M^5 functionally independent of*

$$H = -\frac{1}{\pi} \sum_{i<j} k_i k_j \ln |z_i - z_j| \qquad and \qquad F = \sum_{i<j} k_i k_j |z_i - z_j|^2.$$

Remark 11.3. Chaotic behavior of systems with four vortices was already hinted at by Poincaré in [Poi1]. Numerical evidence of it was discussed by E. Novikov [NovE].

In spite of the fact that the 4-vortex system is generally nonintegrable, the KAM theory guarantees that for any number of vortices there is a set of positive measure in the space of initial conditions for which the motion is *quasiperiodic* [Kha]. Such vortex configurations are organized in the following way: The set of all vortices is split into several groups such that the distances between the groups are much greater than those between the vortices in the groups. In this case the vortex groups interact approximately as single vortices possessing the total circulation. The actual vortex motion is obtained as a superposition of the group motion and the independent vortex motion within the groups.

11.C. Hamiltonian vortex approximations in three dimensions

Just as the vorticity function can be approximated by a collection of point vortices, the vorticity vector field B in $\mathbb{R}^3$ can be taken to be supported on (several) curves.

Note that the corresponding closed two-form $\omega = i_B \mu$, which is the result of contraction of the field B with the volume form μ in $\mathbb{R}^3$, is assumed to be a δ-type differential form, or "current" in the sense of De Rham [DeR]. For the δ-two-form supported on a curve in $\mathbb{R}^3$, the integral over any two-dimensional surface is the algebraic number of intersections of this surface with the supporting curve.

The Euler equation (11.1) defines the evolution law for the vortex curves. Unlike the case of a two-dimensional fluid, the dynamics of such curves still constitute an infinite-dimensional system, though of "much smaller dimension" than the original equation on a smooth vorticity field. The position of every vortex curve is defined by three functions of one variable, while each component of a generic vorticity field in $\mathbb{R}^3$ is a function of three variables.

The dynamics of one smooth vortex curve in $\mathbb{R}^3$ is mathematically very interesting. The first approximation of the vortex motion, where only "local" interaction is considered, turns out to be a completely integrable system. It is known in various contexts under different names: filament equation, ferromagnetic equation, nonlinear Schrödinger equation, Landau–Lifschitz equation for the group $SO(3)$, the Betchov–Da Rios equation, etc. (see the discussion of relations between them in Section VI.3).

The inclusion of the second, already nonlocal, term into the approximation breaks the integrability (see [KlM]).

A more straightforward finite-dimensional model of the Euler equation in three dimensions is an approximation of the velocity and vorticity functions at a finite number of points; see [But, Ose2]. The Clebsch variables provide another way to deal with the canonical Hamiltonian structure in calculations. They are defined on the space that is twice as big as the space of all divergence-free vector fields (or its dual); see [M-W, Zak] and Section VI.2.

11.D. *Finite-dimensional approximations of diffeomorphism groups*

So far, we have been dealing with finite-dimensional models for hydrodynamical systems. However, in a number of two-dimensional cases, the entire group structure behind the fluid dynamics can, in some sense, be approximated as well.

Consider an incompressible fluid on a two-dimensional torus T^2 whose configuration space is the group $S \operatorname{Diff}(T^2)$ of area-preserving (or symplectic) diffeomorphisms of T^2. We show below (following [FZ, FFZ]) that this group can be "approximated" by the groups $SU(n)$ as $n \to \infty$. More precisely, the Lie algebra $S \operatorname{Vect}(T^2)$ admits a continuous deformation known as the family of so-called *sine*-algebras. The latter are infinite-dimensional algebras, and for integral values of the parameter, the finite-dimensional truncations of them turn out to be exactly the algebras $\mathfrak{su}(n)$. The limit of the dual spaces "respects" the Poisson brackets and the structure of Casimir functions, and has been successfully used to approximate the Euler equation in a Hamiltonian way; see [Ze1].

For a two-dimensional torus $T^2 = \{(x_1, x_2) \mod 2\pi\}$, we consider the Lie algebra $S_0 \operatorname{Vect}(T^2)$ of all divergence-free vector fields on the torus with *single-valued* stream functions. The flows generated by those vector fields "do not shift" the total fluid mass. Such stream functions can be assumed to have zero mean. We complexify our Lie algebra, commutator [,], and other operations, and then choose a basis L_k in the form of Fourier exponents $e^{i(k,x)}$, $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus 0$, whose value at a point (x_1, x_2) is $\exp(i(x_1 k_1 + x_2 k_2))$.

The commutators of the basis elements L_k in the Lie algebra $S_0 \operatorname{Vect}(T^2)$ are

$$(11.4) \qquad\qquad [L_k, L_\ell] = (k \times \ell)\, L_{k+\ell},$$

where $k \times \ell = k_1 \ell_2 - k_2 \ell_1$ is the (oriented) area of the parallelogram spanned by k and ℓ; see [Arn16] and Section IV.3.

On the other hand, the commutation relations in the algebras $\mathfrak{sl}(n, \mathbb{C})$ "approximate" those in (11.4) as $n \to \infty$ in the following sense (see [FFZ]). Fix some odd n and consider the following two matrices in $\mathfrak{sl}(n, \mathbb{C})$:

$$F = \operatorname{diag}(1, \varepsilon, \ldots, \varepsilon^{n-1}) \quad \text{and} \quad H = \begin{pmatrix} 0 & 1 & & 0 & 0 \\ & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & \ddots & 1 \\ 1 & 0 & \ldots & & 0 \end{pmatrix},$$

where ε is a primitive nth root of unity and may be taken as, e.g., $\varepsilon = \exp(-4\pi i/n)$. The matrices obey the identities $HF = \varepsilon FH$ and $F^n = H^n = 1$.

Define $n^2 - 1$ matrices J_k, $k = (k_1, k_2) \in \mathbb{Z} \times \mathbb{Z} \pmod{n}$ and $(k_1, k_2) \neq (0, 0) \pmod{n}$, by setting

$$J_{(k_1, k_2)} = \varepsilon^{k_1 \cdot k_2/2} F^{k_1} H^{k_2}.$$

Proposition 11.4. *The matrices J_k have zero trace and span the algebra $\mathfrak{sl}(n, \mathbb{C})$ with the following commutation relations:*

$$(11.5) \qquad [J_k, J_\ell] = 2i \cdot \sin\left(\frac{2\pi(k \times \ell)}{n}\right) J_{k+\ell}.$$

Proof. The proof is an easy calculation. Note that the set of J_k's is closed under composition and inversion:

$$J_k J_\ell = \varepsilon^{-(k \times \ell)/2} J_{k+\ell} \qquad \text{and} \qquad J^{-1}_{(k_1, k_2)} = J_{(-k_1, -k_2)},$$

and all these matrices have determinant equal to 1. $\qquad \square$

As $n \to \infty$ this algebra turns into the algebra $S_0 \, \mathrm{Vect}(T^2)$ of divergence-free vector fields on the torus, with the generators L_k and the relations (11.4) through the identification $(n/4\pi i)\, J_k \mapsto L_k$.

The Euler equation (11.2) on the torus can be approximated by making use of this limit of algebras. First, write (11.2) in terms of the Fourier components of the vorticity $\omega = \sum_m \omega_m e^{i(m,x)}$:

$$(11.6) \qquad \dot{\omega}_m = \sum_k \frac{(m \times k)}{k^2} \omega_k \omega_{m-k}.$$

More generally, we recall that the Euler equation corresponding to a Lie algebra with structure constants C^k_{im} and the inertia tensor a_{ik} in coordinates $\{\omega_i\}$ has the form

$$(11.7) \qquad \dot{\omega}_m = \sum_{k,l,p} a^{kp} C^l_{mp} \omega_k \omega_l$$

on the dual Lie algebra, where a^{kp} is the inverse inertia tensor (see Section 4). The Euler equation (11.6) for the ideal fluid on the torus is reproduced from the latter by setting

$$(11.8) \qquad C^l_{mp} = (p \times m)\delta_{m+p-l,0} \qquad \text{and} \qquad a^{kp} = \frac{1}{k^2}\delta_{k+p,0},$$

with all indices belonging to $(\mathbb{Z} \times \mathbb{Z}) \setminus (0, 0)$.

The $\mathfrak{sl}(n)$-approximations of the divergence-free vector fields on T^2 prompt the introduction of a new dynamical system with the structure constants

$$C^l_{mp} = \frac{n}{2\pi} \sin\left(\frac{2\pi(p \times m)}{n}\right)\delta_{m+p-l,0},$$

with the same metric a^{kp} as in (11.8), and where all index components are now considered modulo n. By imposing a reality condition $\omega_{-m} = \bar{\omega}_m$, one obtains the approximation of the Euler hydrodynamic equation (11.6) on the torus by dynamical systems (11.7) on the algebras $\mathfrak{su}(n)$; see [Ze1].

Remark 11.5 [Ze1]. The above limit of Lie algebras $\mathfrak{sl}(n) \to S_0 \operatorname{Vect}(T^2)$ as $n \to \infty$ respects the structure of Casimir functions on the corresponding spaces. For a given n the algebra $\mathfrak{sl}(n) = \{A \in \operatorname{Mat}(n, \mathbb{C}) \mid \operatorname{tr} A = 0\}$ (or its real form $\mathfrak{su}(n)$) admits $n - 1$ functionally independent Casimir functions, i.e., functions constant on the orbits of the (co)adjoint action:

$$\operatorname{tr} A^2, \operatorname{tr} A^3, \ldots, \operatorname{tr} A^n.$$

In the limit these invariants become the momenta of the corresponding vorticity functions

$$\int_{T^2} \omega^2 \mu, \quad \int_{T^2} \omega^3 \mu, \quad \ldots, \int_{T^2} \omega^n \mu, \ldots$$

(where $\mu = d^2 x$ is the standard area form on the torus), providing an infinite number of Casimirs for the area-preserving diffeomorphism group, and for the two-dimensional Euler equation. On the other hand, in three-dimensional ideal hydrodynamics there is essentially one analytic expression (helicity) for a conserved quantity of Casimir type, which makes the prospects for a reliable group approximation of the 3-D fluid motion rather hopeless.

The infinite-dimensional counterpart of the integrable Euler equation of an n-dimensional rigid body (see [Man], or Section VI.1.B) was obtained in [War] by considering the limit of the algebras $\mathfrak{so}(n)$ as $n \to \infty$. For the Euler equation on the 2-D sphere, an interesting model involving the rich representation theory of the dodecahedral group has been studied in [VshS].

Remark 11.6. The algebra (11.5) is the nonextended part (also called the "cyclotomic family") of an infinite-dimensional *sine-algebra* [Hop, FFZ, FZ] with an *infinite* number of generators J_k, $k = (k_1, k_2) \in \mathbb{Z} \times \mathbb{Z}$ and $(k_1, k_2) \neq (0, 0)$, and the commutation relations

$$[J_k, J_\ell] = 2i \cdot \sin\left(\frac{2\pi (k \times \ell)}{\lambda}\right) J_{k+\ell} + (a \cdot k)\delta_{k+\ell,0}.$$

Here the constant λ is not necessarily an integer, but it is now an arbitrary complex number; $a = (a_1, a_2)$ is a fixed plane vector; the notation $\delta_{k+\ell,0}$ stands for 1 if $k = -\ell$ and 0 otherwise; and $(a \cdot k) := a_1 k_1 + a_2 k_2$. The term $(a \cdot k)\delta_{k+\ell,0}$ defines a nontrivial extension of the sine-algebra. We refer to [FFZ, Rog, KLR] for the definition and discussion of such extensions. Here we merely mention that in the limit $\lambda \to \infty$, after a suitable renormalization, this extension corresponds to introducing multivalued (nonperiodic) stream functions x_1 or x_2 on the torus $T^2 = \{(x_1, x_2) \bmod 2\pi\}$ whose flows are univalued periodic vector fields on T^2.

The recent interest in the sine-algebras is not only due to their hydrodynamical applications. Viewed as deformations of the Poisson algebra of functions on a two-dimensional torus, they are related to the Moyal product of functions on a linear symplectic space $\mathbb{R}^{2n}$ [Moy], the algebras of differential and pseudodifferential

operators of one and several variables, the algebra of q-analogues of pseudodifferential operators [KLR], and the algebras with continuum root systems [SaV].

§12. The Navier–Stokes equation from the group viewpoint

The Euler equation of ideal hydrodynamics,

$$\dot{v} = -(v, \nabla)v - \nabla p \qquad (\text{or} \quad \dot{\omega} = -\{v, \omega\}, \quad \omega = \text{curl } v),$$

is related to the Navier–Stokes equation of a viscous fluid,

$$\dot{v} = -(v, \nabla)v - \nabla p + f + \nu \Delta v \qquad (\text{or} \quad \dot{\omega} = -\{v, \omega\} + \text{curl } f + \nu \Delta \omega),$$

in the same way as the classical Euler equation of a rigid body,

$$\dot{m} = m \times \omega,$$

is associated to a more general equation, involving friction and external angular momentum,

$$(12.1) \qquad \qquad \dot{m} = m \times \omega + F - \nu m.$$

Here the "friction operator" ν is symmetric and positive definite. The distributed mass force f, which appeared in the Navier–Stokes equation, is similar to the external angular momentum F, and it is the origin of the fluid motion. The viscous friction $\nu \Delta v$ is analogous to the term $-\nu m$ in (12.1) slowing the rigid body motion.

The similarity becomes especially noticeable if one (following V.I. Yudovich, 1962) rewrites the equations in components in the eigenbasis of the friction operator. For example, for the Navier–Stokes equation with periodic boundary conditions one can expand the vorticity field and the force f into the ordinary Fourier series. The equations in both of the cases have the following form:

$$(12.2) \qquad \qquad \dot{x}_i = \sum a_{ijk} x_j x_k + \sum f_i - \nu_i x_i.$$

In practice, one usually considers a *Galerkin approximation* in which only a finite number of terms is kept.

The first term corresponds to the Euler equation and describes the inertia motion. It follows from the properties of the Euler equation that the divergence of this term is equal to zero. Furthermore, the Euler equation of an ideal fluid in any dimension, as well as that of a rigid body, has a quadratic positive definite first integral, the kinetic energy. Therefore, for $f = \nu = 0$ the vector field on the right-hand side of equation (12.2) is tangent to certain ellipsoids centered at the origin. This implies that during the evolution defined by this equation, at least in the finite-dimensional situation, there is neither growth nor decay of solutions (in the energy metric).

The term corresponding to the friction dominates over the constant "pumping" f when considered sufficiently far away from the origin. Hence, in that remote region, the motion is directed towards the origin, and an infinite growth of solutions is impossible (provided that the problem is finite-dimensional).

Since the "pumping" f pushes a phase point out of any neighborhood of the origin, while the friction returns it from a distance, a motion in the system of a rigid body (12.1) approaches an intermediate regime-attractor. For instance, this attractor can be a stable stagnation point or a periodic motion, while for sufficiently high dimension of the phase space it can appear to be a "chaotic" motion sensitive to the initial condition.

If the friction (or viscosity) coefficient ν is high enough, then the attractor will necessarily be a stable equilibrium position. While the parameter ν decreases (i.e., the reciprocal parameter, the Reynolds number $Re := 1/\nu$, increases), bifurcations of the equilibrium are possible, and the attractor can become a periodic motion and later a "stochastic" one.

The hypothesis that this mechanism is responsible for the phenomenon of turbulenization of a fluid motion for large Reynolds numbers has been suggested by many authors. In particular, in the Spring of 1965 A.N. Kolmogorov spelled it out at a meeting of the Moscow Mathematical Society, during a discussion of the talk by N.N. Brushlinskaya on bifurcations in equation (12.1) [Bru]. Also in 1965, the first author, in his talk on this theory in R. Thom's seminar IHES, formulated the conjecture that negativity of curvatures of the diffeomorphism group implies instability of fluid motion for the Euler dynamics, as well as for the corresponding attractors in the Navier–Stokes equation (see [Arn11,18]).

To normalize the attractor, A.N. Kolmogorov suggested considering the "pumping" proportional to the same small parameter ν as viscosity, and he formulated the following two conjectures for the latter case.

(1) *The weak conjecture:* The maximum of the dimensions of minimal attractors[1] in the phase space of the Navier–Stokes equations (as well as of their Galerkin approximations (12.2)) grows along with the Reynolds number $Re = 1/\nu$.

(2) *The strong conjecture:* Not only maximum, but also the minimum of the dimensions of the minimal attractors mentioned above increases with Re.

Both of these hypotheses, with respect to two-dimensional as well as three-dimensional hydrodynamics, still remain open.

In 1963 E. Lorenz [Lor] studied the following system in the three-dimensional phase space,

$$\begin{cases} \dot{x} = -10x + 10y, \\ \dot{y} = rx - y - xz, \\ \dot{z} = -\frac{8}{3}z + xy, \end{cases}$$

and numerically discovered an attractor with exponentially unstable motion along it for $r = 28$. This phenomenon has been called a *strange attractor*, and later it was investigated in many numerical–analytical, as well as theoretical, papers (e.g., see references in [PSS]). The above system exhibits varied and interesting properties

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[1]An attractor is called a minimal attractor if it does not contain smaller attractors.

for different r. For instance, as the parameter r decreases from 100.795 to 99.524 one observes an infinite sequence of bifurcations of period doubling of a stable periodic orbit, analogous to the successive period doublings in the Feigenbaum family of maps of a segment.

It is interesting to observe that the Lorenz model is similar to the Galerkin approximations of the Navier–Stokes equations (12.2).

For the Galerkin system (12.2) the domain where the energy grows is bounded by some ellipsoid in the phase space. Outside of that ellipsoid the energy decreases, and a phase point starts returning to the origin.

For the Lorenz system, the role of energy is played by a nonhomogeneous quadratic function. The instability in the Lorenz model is apparently stronger than in the Kolmogorov one. One can check how the motion along the Lorenz strange attractor sensitively depends on the initial conditions, while for the Kolmogorov model it remains a conjecture. It is proven only that a stationary flow indeed loses stability as the Reynolds number increases. The case of the sine profile $(\sin y)\, \partial/\partial x$ of the exterior force on a two-dimensional torus has been settled in [MSi]. The bifurcations in the Kolmogorov model has been studied by Yudovich, who proved the existence of a secondary regime, as well as the long-wave instability of more general steady shear flows $u(y)\, \partial/\partial x$ [Yu4].

A.N. Kolmogorov always emphasized that preservation of stability of a steady flow, even for the infinitely growing Reynolds number, would not contradict hydrodynamical experiments, under the assumption that the basin of the corresponding attractor shrinks fast enough.

The idea of a connection between the theory of hydrodynamical instability and the study of stochastization in ergodic theory of dynamical systems was repeatedly suggested by A.N. Kolmogorov for several years. For instance, in the program of his 1958/1959 seminar, which was posted on the bulletin board of the Department of Mechanics and Mathematics (Mech–Mat) at Moscow State University, he listed the following themes:

1. Boundary value problems for hyperbolic equations whose solutions everywhere depend discontinuously on a parameter (see, for example, [Sob2]*).*

2. Problems on classical mechanics in which the eigenfunctions depend everywhere discontinuously on a parameter (a survey of these problems is contained in a lecture by Kolmogorov at the Amsterdam Congress in 1954).

3. Monogenic Borel functions and quasianalytic Gonchar functions (in the hope of applications to problems of type 1 and 2).

4. The rise of high-frequency oscillations when the coefficients of the higher derivatives tend to zero (papers of Volosov and Lykova for ordinary differential equations).

5. In the theory of partial differential equations with a small parameter at the higher derivatives, there has recently been a study of phenomena of boundary layers and interior layers converging to surfaces of discontinuity of limiting solutions, or of their derivatives, as viscosity vanishes. In real

*turbulence the solutions deteriorate on an everywhere dense set. The math-
ematical study of this phenomenon is assumed to be carried out at least on
model equations (the Burgers model?).*

*6. Questions of stability of laminar flows. Asymptotically vanishing stabil-
ity (at least on model equations).*

*7. Discussion of the possibility of applications to some problems in real
mechanics and physics of the ideology of the metrical theory of dynamical
systems. Questions of stability of various types of spectrum. Structurally sta-
ble systems and structurally stable properties (in the latter direction, hardly
anything is known for systems with several degrees of freedom!).*

*8. Consideration (at least on models) of the conjecture that, in the situation
at the end of 5 above, in the limit the dynamical system turns into a random
process (the conjecture of the practical impossibility of a long-term weather
forecast).*

Constructions of the modern theory of dynamical systems, such as the
Kolmogorov–Sinai entropy [Kol, Si1] measuring the degree of stochastization of
a deterministic dynamics, were undertaken specifically to develop this program.

In 1970 Ruelle and Takens formulated the conjecture that turbulence is the ap-
pearance of attractors with sensitive dependence of motion on the initial conditions
along them in the phase space of the Navier–Stokes equation [R-T]. In spite of
the vast popularity of this paper, even the existence of such attractors still remains
an open question (not to mention the earlier hypotheses of Kolmogorov on the
growth in dimension of the minimal attractors).

Infinite-dimensionality of the phase space of the Navier–Stokes equation affects
the foundation of the passage to the system (12.2) and to Galerkin approximations
as follows. The friction operator in hydrodynamical problems is the product of
viscosity ν and the Laplace operator. The absolute values of its eigenvalues ν_i
increase with the order of the corresponding harmonics. Hence, the high harmonics
rapidly decay for nonvanishing viscosity. This implies that a phase point of the
infinite-dimensional space is attracted to the finite-dimensional one, where the
coordinates are the amplitudes of the lower harmonics; see [MPa, FoT, D-O].
For a fixed viscosity the analysis of the Galerkin approximation allows one, in
principle, to draw conclusions on the behavior of the actual solutions (see, e.g.
[MatS]).

However, if we are interested in solution behavior as viscosity (the coefficient
ν at the Laplace operator) goes to zero, then one has to consider the number of
harmonics (in the Galerkin approximation) rapidly increasing as $Re = 1/\nu \to \infty$.
The first explicit estimate of the Hausdorff dimension of the maximal attractor A
of the Navier–Stokes equation for the case of the two-dimensional torus dim $A \leq$
const $\cdot \, \nu^{-4}$, given in [Ilsh], has been substantially improved. The best current
majorant of this number is

$$\dim A \leq \frac{1}{\pi} \frac{\|f\|_{L^2} \cdot \mathrm{vol}(M)}{\nu^2}$$

(where f is the external force, M is a domain of finite volume, and the boundary condition is zero). It was obtained by A. Ilyin [Ily], based on [CFT]. (We refer to [B-V, Tem] for the contemporary state of the art.) As the dimension of the physical space grows, so does the number of harmonics, corresponding to the eigenvalues whose magnitude is smaller than a given number. It follows that the Galerkin approximation is to be of greater size.

The character of the first, inertia, term in (12.2) changes drastically in the passage from two-dimensional fluid flows to three- (or higher-) dimensional ones. The reason lies in the distinctions among the geometries of the coadjoint orbits of the corresponding diffeomorphism groups (or the absence of invariants of enstrophy-type for the higher-dimensional Euler equations; see Sections 9 and 11). Further, this geometry also obstructs a better foundation for the correspondence between the Galerkin approximation and the original Navier–Stokes equation in the three-dimensional case.

In the sixties most specialists in partial differential equations (with the notable exception of V.I. Yudovich) regarded the lack of global existence and uniqueness theorems for solutions of the Navier–Stokes equation as the explanation of turbulence. This point of view was never popular among physicists.

For the three-dimensional Navier–Stokes equation for small or vanishing viscosity, the existence and uniqueness theorems for an arbitrarily large period of time are still open questions.

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Chapter II

Topology of Steady Fluid Flows

Cold and warm ocean currents (for instance, the Gulf Stream) determine the climate of continents beyond the reach of human intervention. The power of the currents' influence is due to their permanentness and stability. In this chapter we are going to study the corresponding idealized model of steady flows of an incompressible fluid. Such flows are stationary solutions of the Euler equation, and they have very peculiar topology and existence conditions. They often turn out to be "attractors" in phase space of the viscous Navier–Stokes equation. In this case the structure of such flows might give an "approximate picture" of an arbitrary fluid motion after a long period of time.

§1. Classification of three-dimensional steady flows

1.A. Stationary Euler solutions and Bernoulli functions

In this chapter we will be dealing with solutions of the Euler equation that do not depend on time.

Definition 1.1. An ideal *steady* (or *stationary*) incompressible fluid flow $v(x)$ in a domain $M \subset \mathbb{R}^n$ is a divergence-free solution ($\operatorname{div} v = 0$) of the *stationary Euler equation*

$$0 = -(v, \nabla)v - \nabla p,$$

for some pressure function p on M.

The same equation in the form $-\nabla_v v - \nabla p = 0$ for a velocity field satisfying $L_v \mu = 0$ is valid for an arbitrary n-dimensional Riemannian manifold M with measure μ.

For the three-dimensional case ($n = 3$), a virtually complete description of analytic stationary flows is given by the following theorem:

Theorem 1.2 [Arn 3, 4, 16]. *Assume that the region $M \subset \mathbb{R}^3$ is bounded by a compact analytic surface, and that the field of velocities is analytic and not*

everywhere collinear with its curl. Then the region of the flow can be partitioned by an analytic submanifold into a finite number of cells, in each of which the flow is constructed in a standard way. Namely, the cells are of two types: those fibered into tori invariant under the flow and those fibered into surfaces invariant under the flow, diffeomorphic to the annulus $\mathbb{R} \times S^1$ (see Fig. 9). On each of these tori the flow lines are either all closed or all dense, and on each annulus all flow lines are closed.

(a)   (b)
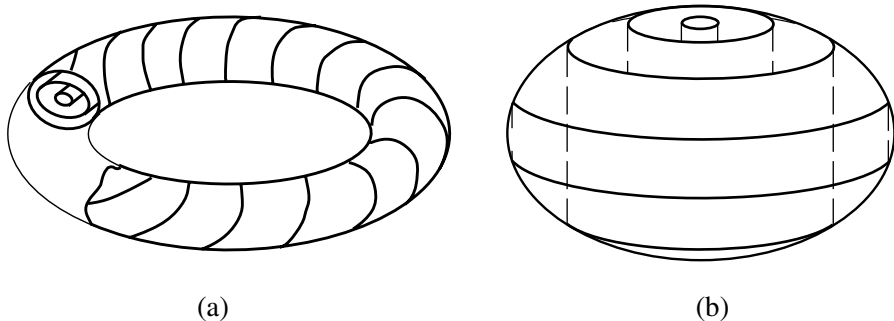
FIGURE 9.   Regions of a steady flow fibered (a) into tori and (b) into annuli.

The stationary Euler equation $(v, \nabla)v = -\nabla p$ in $M \subset \mathbb{R}^3$ can be rewritten as

$$v \times \operatorname{curl} v = \nabla \alpha$$

for the function $\alpha = p + \frac{\|v\|^2}{2}$.

Definition 1.3. The function $\alpha : M \to \mathbb{R}$ defined by the relation $v \times \operatorname{curl} v = \nabla \alpha$ (modulo an additive constant) is called the *Bernoulli function* of the steady flow v.

By the very definition, the velocity field v, as well as the vorticity field $\operatorname{curl} v$, is tangent to the level surfaces of the Bernoulli function α. In other words, α is the first integral of the flow defined by the field v in the domain M.

Note that the stationary three-dimensional Navier–Stokes equation (describing a viscous incompressible fluid) generically does not admit any nontrivial first integrals [Ko3].

Remark 1.4. In invariant terms the stationary Euler equation

$$L_v u = -dp$$

is equivalent to $i_v du + d i_v u = -dp$, or to the equation

$$i_v du = -d\alpha \qquad \text{for} \quad \alpha = p + i_v u.$$

The invariance of α (i.e., $L_v \alpha = 0$) follows from the relation $i_v d\alpha = 0$.

Note that the condition $v \times \text{curl } v = \nabla \alpha$ in $\mathbb{R}^3$ can be reformulated in the form valid for any manifold M: The vector fields v and $\text{curl } v$ commute ($\{v, \text{curl } v\} \equiv 0$). To verify this for a three-dimensional Riemannian manifold M, one employs the following formula of vector calculus:

$$(1.1) \qquad \text{curl}(\eta \times \xi) = \{\xi, \eta\} + \eta(\text{div } \xi) - \xi(\text{div } \eta)$$

on any three-dimensional Riemannian manifold. (Here $(\eta \times \xi)$ is the vector field dual to the 1-form $i_\xi i_\eta \mu$ on M: $(i_\xi i_\eta \mu)\zeta = \mu(\eta, \xi, \zeta) = (\eta \times \xi, \zeta)$.) By taking vorticity of both sides of $v \times \text{curl } v = \text{grad } \alpha$, we obtain $\{v, \text{curl } v\} \equiv 0$.

The classification theorem above relies on the following observation about the structure of α-level surfaces for a three-dimensional manifold M.

Proposition 1.5. *Every noncritical level surface of α that does not intersect the boundary of M^3 is diffeomorphic to a torus. For appropriate variables $(\varphi_1, \varphi_2 \mid \text{mod } 2\pi)$ and z in a neighborhood of such a torus both fields v and $\xi = \text{curl } v$ have constant components*

$$v = v_1(z)\frac{\partial}{\partial \varphi_1} + v_2(z)\frac{\partial}{\partial \varphi_2}, \qquad \xi = \text{curl } v = \xi_1(z)\frac{\partial}{\partial \varphi_1} + \xi_2(z)\frac{\partial}{\partial \varphi_2},$$

along the torus with angular coordinates (φ_1, φ_2), while z indexes the tori.

The coordinates φ_1, φ_2, z are analogues of the action-angle variables of classical mechanics. The theorem means, in particular, that the field lines of both v and $\text{curl } v$ lie on the tori $\alpha = \text{const}$. These lines on a given torus are either closed (if the ratio of the frequencies v_2/v_1 for the field v, respectively, ξ_2/ξ_1 for the field $\xi = \text{curl } v$, is rational) or dense. The proof is given in Section 1.B.

Remark 1.6. In the case of $\alpha \equiv \text{const}$ (all α-levels are critical), the fields v and $\text{curl } v$ are collinear at each point ($v \times \text{curl } v \equiv 0$). Such fields are called *force-free* fields in magnetohydrodynamics.

If a force-free field v is nowhere zero, then $\text{curl } v = \varkappa \cdot v$, where the "ratio" $\varkappa : M \to \mathbb{R}$ is a smooth function. The function $\varkappa$ is a first integral of the field v (as well as of the field $\text{curl } v$). Indeed, $0 \equiv \text{div}(\text{curl } v) = \text{div } \varkappa \cdot v = (\text{grad } \varkappa, v)$. Hence, every connected component of a nonsingular level surface of $\varkappa$ is a torus, since such a surface is oriented and it admits a nonvanishing tangent vector field v (the same reasoning is used in the proof of Proposition 1.5; see Section 1.B). The field lines of v are windings on these tori (in the corresponding coordinates φ_1, φ_2, z, the frequency ratios $\dot{\varphi}_1/\dot{\varphi}_2 = \kappa(z)$ will be constant along the field lines of v). Therefore, even in the case of a force-free field, the field lines lie on two-dimensional tori, provided that the field does not have zeros and the function $\varkappa$ is not constant.

A force-free field v with $\text{curl } v = \lambda v$, where λ is a constant (i.e., an eigenfield v of the curl operator), can have a much more complicated topology.

Definition 1.7. The eigenfields of the operator "curl" are called *Beltrami* fields.

Corollary 1.8. *If a steady analytic flow has a trajectory that is not contained in any analytic (singular) surface, then the flow is defined by a Beltrami field.*

Indeed, non-Beltrami flows enjoy a first integral (either the Bernoulli function α or the ratio function $\varkappa$).

Example 1.9. On the three-dimensional torus $\{(x, y, z) \mid \mod 2\pi\}$, a family of Beltrami fields is given by the so-called ABC flows

$$
\begin{cases}
v_x = A \sin z + C \cos y, \\
v_y = B \sin x + A \cos z, \\
v_z = C \sin y + B \cos x.
\end{cases}
$$

The divergence-free vector fields of this three-parameter family are eigen for the vorticity operator curl $v = v$. The ABC flows have been discovered by Gromeka in 1881, rediscovered by Beltrami in 1889, and proposed for study in the present context in [Arn4, Chi1] (see the references and details in [VasO]).
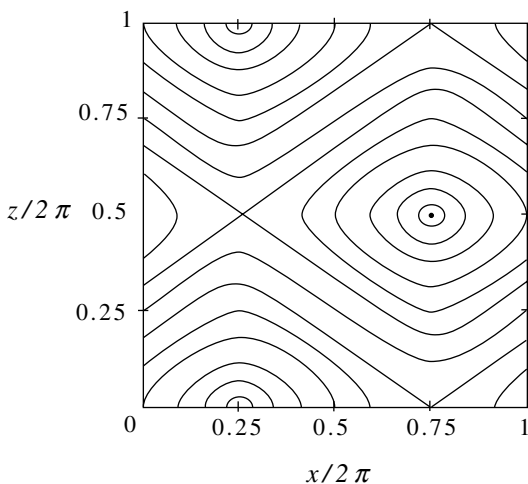
FIGURE 10.   The projection of the streamlines on the (x, z)-plane in the integrable case $C = 0$ (see [Dom]). The field components do not depend on y.

When one of the parameters A, B, or C vanishes, the flow is integrable (Fig. 10). Perturbation techniques used in the near-integrable cases allows one to predict strong resonances (see discussion and results of numerical simulations in [Dom]). For such perturbations some tori filled out by field lines (magnetic surfaces) persist (see, e.g., [AKN]), whereas others are disrupted, leading to regions with chaotic behavior of trajectories. There is numerical evidence that certain trajectories densely

fill three-dimensional domains (Fig. 11). In particular, the search for integrable cases, carried out in [Dom] by studying complex-time singularities of field trajectories, showed (numerically) the absence of integrability for $ABC \neq 0$. For the case $A = \sqrt{3}, B = \sqrt{2}, C = \sqrt{1}$ see [Hen], while the more general situation was treated in [Dom]. The absence of meromorphic integrals for generic ABC flows with $A = B$ and for the ABC flows with $0 \neq A \neq B \neq 0$ and small $C \neq 0$ has been proven by Ziglin [Zig2].
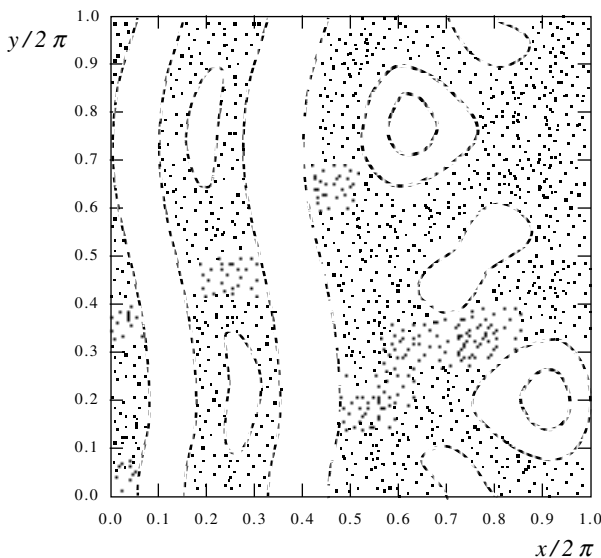
FIGURE 11.   A typical Poincaré section for the ABC flows ($A^2 = 1$, $B^2 = \frac{2}{3}$, and $C^2 = \frac{1}{3}$). Some field lines seem to fill three-dimensional regions ([Hen] or [Dom]).

A similar study of field symmetries and stagnation points for an analogue of the ABC flow in a three-dimensional ball can be found in [Zhel].

Note that if the field v satisfying curl $v = \varkappa \cdot v$ is not divergence free, then the topological properties of its trajectories are different from those discussed here: The flow is generically nonintegrable even for a nonconstant function $\varkappa : M \to \mathbb{R}$ (see [MYZ]).

1.B.   Structural theorems

We first prove a smooth analogue of (real-analytic) Theorem 1.2 for a closed manifold.

Let α be the Bernoulli function for a steady flow v on an orientable 3-dimensional manifold M without boundary. Denote by $\Gamma \subset M$ the preimage of the critical values of α.

Theorem 1.10 (=1.5′). *Every connected component of the set $M \setminus \Gamma$ is fibered into two-dimensional tori invariant under the flow of v. The motion on each torus is quasiperiodic (the field lines are either all closed or all dense).*

Proof. The function α is the first integral for the vector fields v and $\xi := \operatorname{curl} v$. Since these fields commute, their flows give rise to an $\mathbb{R}^2$-action on every level surface of α. Each noncritical α-level is a smooth closed surface, and hence it is a torus or a Klein bottle. (In other words, the Euler characteristic of any noncritical α-level is zero: If $\nabla\alpha \neq 0$, then the velocity field v provides an example of a tangent vector field nonvanishing on the surface.) Furthermore, this surface is cooriented by $\nabla\alpha$. As a result, we see that the surface is orientable; i.e., it is a torus.

On each α-level the flow of ξ acts transitively on integral curves of v, and thus the latter are either all closed or all dense in the level surface. In the coordinates on a torus in which the $\mathbb{R}^2$-action is given by linear translations, the fields v and $\operatorname{curl} v$ become the vector fields with constant coefficients. $\square$

We now turn to the real-analytic theorem (we follow the exposition in [GK2]).

Definition 1.11. A subset of a real-analytic manifold is called *semianalytic* if locally it may be defined by a finite number of real-analytic equations and inequalities.

We will need certain properties of such sets summarized in the following

Lemma 1.12. *Let M and N be compact connected real-analytic manifolds (possibly with boundary) and $f : M \to N$ a real-analytic map. Then*

(i) *Any semianalytic subset X of M divides M into a finite number of connected components.*

(ii) *The image $f(X)$ is a semianalytic subset of N, provided that $\dim N \leq 2$.*

(iii) *Assume that the rank of f is equal to $\dim N$ at at least one point of M, and Y is a nowhere dense semianalytic subset of N. Then the preimage $f^{-1}(Y)$ is semianalytic and nowhere dense in M.*

Proof. Assertions (i) and (ii) are classical results due to Łojasiewicz [Łoj]. To prove (iii) consider the set K of critical points of f. The set $f^{-1}(Y) \cap (M \setminus K)$ is nowhere dense because the restriction of f to $M \setminus K$ is a submersion. Since $\operatorname{rank} f = \dim N$ somewhere on M, the set K is, in turn, nowhere dense in M. Thus $f^{-1}(Y)$ is nowhere dense, for it is the union of two sets, each of which is nowhere dense. It is clear by definition that $f^{-1}(Y)$ is semianalytic. $\square$

Proof of Theorem 1.2. Suppose first that M is a connected manifold without boundary ($\partial M = \emptyset$). Assume also that all the data (the volume form, the metric, and the velocity field v) are real-analytic. In this case one claims that $U = M \setminus \Gamma$

has a finite number of connected components, and each of them is fibered into two-dimensional tori invariant under the flow.

Indeed, under the hypothesis of the theorem, the map $\alpha : M \to \mathbb{R}$ is analytic, and we can take $f = \alpha$. As above, let K be the critical set of α. Then $\alpha(K)$ is semianalytic by (ii) and nowhere dense by the Sard lemma. Therefore by (iii), $\Gamma = \alpha^{-1}(\alpha(K))$ is semianalytic and nowhere dense in M. Applying (i) to $X = \Gamma$, we see that U is dense in M, and U has a finite number of connected components.

To complete the proof for M without boundary, it suffices to apply Theorem 1.10.

Consider now the case of M with boundary ($\partial M \neq \emptyset$). Again, let K be the critical set of α and C the critical set of $\alpha\big|_{\partial M}$. As above, the union Y of the sets $\alpha(K)$ and $\alpha(C)$ is a semianalytic set nowhere dense in $\mathbb{R}^2$. Therefore, $\Gamma = \alpha^{-1}(Y)$ is nowhere dense, semianalytic, and invariant with respect to the flow.

Although we may not have an $\mathbb{R}^2$-action now, since M is a manifold with boundary, we do have a local $\mathbb{R}^2$-action on $M \setminus \partial M$. Furthermore, the maps $\alpha\big|_U$ and $\alpha\big|_{\partial M \cap U}$ are still proper submersions onto their images. Consider the orbit $\mathcal{O}_x$ through a point $x \in U$ of the local $\mathbb{R}^2$-action. The same argument as in the proof of Theorem 1.10 shows that $\mathcal{O}_x$ is either a torus or an annulus. In the former case the integral curves of v are all closed or all dense on $\mathcal{O}_x$. Observe that $L = \mathcal{O}_x \cap \partial M$ is invariant under the flow of v, and thus $\mathcal{O}_x$ is an annulus if and only if it meets ∂M. By the definition of U, the field ξ is transversal to ∂M along L. This implies that L is the union of two closed integral curves of v. Since we have a locally well-defined $\mathbb{R}^2$-action, all the v-streamlines on $\mathcal{O}_x$ must be closed.

Let U_0 be a connected component of U. The orbits $\mathcal{O}_x, x \in U_0$, are either all tori or all annuli. Indeed, for all $x \in U$ the levels $F_x = \alpha^{-1}(\alpha(x))$ are transversal to ∂M, and hence the connected components $\mathcal{O}_x$ of F_x are diffeomorphic to each other for all $x \in U_0$. Theorem 1.2 is proved. $\qquad\square$

§2. Variational principles for steady solutions and applications to two-dimensional flows

2.A. *Minimization of the energy*

Consider the following variational problem (which *a priori* is not related to the stationary Euler solutions). Let M be a three-dimensional closed Riemannian manifold equipped with a volume form μ, and ξ a divergence-free vector field on M. The *energy* of the field is the integral

$$E = \frac{1}{2}\langle \xi, \xi \rangle = \frac{1}{2}\int_M (\xi, \xi)\mu.$$

Problem 2.1. Find the minimum energy and the extremals among all fields obtained from a given field ξ by the action of volume-preserving diffeomorphisms of the manifold M.

Here the action of a volume-preserving diffeomorphism $g : M \to M$ associates to a divergence-free field ξ on M another divergence-free field $g_*\xi$ such that the flux of the field ξ across any surface σ is equal to the flux of $g_*\xi$ across $g(\sigma)$. In other words, the field is frozen into an incompressible fluid filling M: The vector field can be thought of as drawn on the elements of the fluid and expanding as these elements expand.

In the case of the manifold M with boundary ∂M, the field ξ is assumed to be tangent to ∂M, and the diffeomorphisms send the boundary ∂M into itself.

In the next chapter we will be concerned with the energy minimum and explicit estimates on it in terms of the field topology. Here we deal exclusively with the topology of the extremal fields.

Theorem 2.2 (see, e.g., [Arn9]). *The extremals of the problem stated above are the divergence-free vector fields that commute with their vorticities. In particular, they coincide with the steady Euler flows in M.*

Proof. Let η be any divergence-free field on M. The variation $\delta\xi$ of a field ξ under the infinitesimal diffeomorphism defined by η is given by the Lie bracket $\delta\xi = [\eta, \xi] = \{\xi, \eta\}$ (in coordinate form the Poisson bracket of the vector fields ξ and η is $\{\xi, \eta\} = (\xi, \nabla)\eta - (\eta, \nabla)\xi$).

Consequently, the variation of the energy is $\delta E = \langle \xi, \delta\xi \rangle = \langle \xi, \{\xi, \eta\} \rangle$. Assume that the vector field ξ is extremal for the energy functional.

By formula (1.1)—$\mathrm{curl}(\eta \times \xi) = \{\xi, \eta\} + \eta(\mathrm{div}\,\xi) - \xi(\mathrm{div}\,\eta)$—which is valid on any three-dimensional Riemannian manifold, and by the divergence-free property for the fields ξ and η, one can rewrite the energy variation at the extremal field ξ as

$$0 = \delta E = \langle \xi, \mathrm{curl}(\eta \times \xi) \rangle = \langle \mathrm{curl}\,\xi, (\eta \times \xi) \rangle = \langle \eta, (\xi \times \mathrm{curl}\,\xi) \rangle.$$

Since η is divergence free, the cross product $\xi \times \mathrm{curl}\,\xi$ is orthogonal to all divergence-free fields. Therefore, it is a gradient: $\xi \times \mathrm{curl}\,\xi = \mathrm{grad}\,\alpha$, whence, by taking the curl of both sides we obtain $\{\xi, \mathrm{curl}\,\xi\} \equiv 0$, as required. $\square$

Remark 2.3. In the case of a two-dimensional manifold M, we obtain the equation

$$\nabla u \times \nabla \Delta u \equiv 0$$

on the stream function u of the extremal field $\xi = \mathrm{grad}\,u$. This equation says that the gradient of the extremal function is collinear with that of its Laplacian (see Section 2.C).

The above result is valid not only for smooth vector fields ξ, but it holds also in a weaker form of the integral identity $\langle \eta, (\xi \times \mathrm{curl}\,\xi) \rangle = 0$, provided that a minimizer ξ exists. Note that existence of smooth and nonsmooth extremals in this problem is a very subtle question. We refer to [Bur, ATL] (see also Sections 2 and 6 below) for existence theorems (of, generally speaking, nonsmooth minimizers)

in the two-dimensional case. For dimension greater than 2, there is no proof that the extremals exist except for certain partial results (cf. [L-A, LS4, Vai, GK2]).

Remark 2.4. A similar calculation leads to the following expression for the second variation of the energy:

$$\delta^2 E = \langle \{\xi, \eta\}, \{\xi, \eta\} \rangle + \langle \{\xi, \eta\}, ((\operatorname{curl} \xi) \times \eta) \rangle,$$

where ξ is an extremal field whose first and second variations are given by the Taylor formula

$$(2.1) \qquad \xi(\varepsilon) = \xi + \varepsilon \{\xi, \eta\} + \frac{\varepsilon^2}{2} \{\{\xi, \eta\}, \eta\} + \cdots, \qquad \varepsilon \to 0,$$

in terms of a divergence-free vector field η.

Remark 2.5. The Taylor series (2.1) for $\xi(\varepsilon)$ is obtained while solving the ordinary differential equation on $\xi(t)$,

$$\frac{d\xi(t)}{dt} = \{\xi(t), \eta\},$$

by substituting the series

$$\xi(t) = \xi + t\xi_1 + \frac{t^2}{2!}\xi_2 + \cdots.$$

The field $\xi(\varepsilon)$ is obtained from ξ by the action of the phase flow transformation of η corresponding to a small time interval ε.

All the fields that can be obtained from ξ by the action of volume-preserving diffeomorphisms form a submanifold in the vector space of all divergence-free vector fields, that is, the orbit of the point ξ. The tangent affine subspace to this "smooth" submanifold at the point ξ is formed by the vectors $\xi + \{\xi, \eta\}$ with arbitrary divergence-free η's.

To calculate the second differential of a function on a submanifold of a vector space at a point *it is not enough* to calculate the second differential of the restriction of the function to the affine subspace tangent to the submanifold at this point. The genuine second differential of the restriction of the function to the *submanifold* and the second differential of the restriction of the same function to the *affine tangent space* at a critical point (of the function restriction to this submanifold) are two *different* quadratic forms on the tangent space. (Here we consider the tangent space as the vector space centered at the critical point.)

Formula (2.1) defines the mapping of a domain of "small" vector fields $\varepsilon\eta$ to the orbit of the field ξ. The energy of the image field, considered as a function of the field $\varepsilon\eta$, is the functional on the vector space of divergence-free vector fields $\{\varepsilon\eta\}$.

The first variation of this functional vanishes if ξ is a critical point of the restriction of the energy to the orbit. Its second variation $\delta^2 E$ is given by the above formula (as a quadratic form of $\varepsilon\eta$).

Proposition 2.6. *If ξ is a critical point of the restriction of the energy to the submanifold, the value of the second variation quadratic form depends only on the tangent vector $\zeta = \{\xi, \varepsilon\eta\}$, and it* does not *depend on the particular choice of the field η.*

Proof. We can replace η by a field $\eta + u$ where $\{\xi, u\} = 0$ (otherwise ζ would change). The contribution of u to the quadratic term in series (2.1) is then $\frac{\varepsilon^2}{2}w$, where $w = \{\{\xi, \eta\}, u\}$. Since $\{\xi, u\} = 0$, we get from the Jacobi identity that $w = -\{\{\eta, u\}, \xi\}$. The latter vector is tangent to the orbit at ξ. Hence the first variation of the energy is vanishing on this vector w. Adding the vector $\frac{\varepsilon^2}{2}w$ to the vector $\xi(\varepsilon)$ (given by (2.1) and being at a distance of order ε from ξ) we change the value of the energy by a quantity of order ε^3. Thus the addition of u to η contributes nothing to the quadratic part of the Taylor series of the energy restriction to the orbit of ξ (provided that the vector field ξ is a critical point). $\square$

2.B. The Dirichlet problem and steady flows

The energy minimization Problem 2.1 acquires the following form of the Dirichlet problem in the two-dimensional case. Let M be a two-dimensional Riemannian manifold (possibly with boundary) with a Riemannian volume form μ.

Problem 2.1′. Find the infimum and the minimizer of the *Dirichlet integral*

$$E(u) = \frac{1}{2} \int_M (\nabla u, \nabla u)\mu$$

among all the smooth functions u (on the manifold M) that can be obtained from a given function u_0 by the action of area-preserving diffeomorphisms of M to itself.

In order to see that this is the two-dimensional counterpart of Problem 2.1, one can consider the skew gradient sgrad u instead of the true gradient ∇u (on which the functional E has, of course, the same value). Then u is regarded as a Hamiltonian function, whose definition is invariant: Any area-preserving change of coordinates for the function u implies the corresponding diffeomorphism action on the field sgrad u.

For instance, let M be the disk $x^2 + y^2 \le 1$, and let u_0 be a function that vanishes at the boundary and has only one critical point (for instance, a maximum) in the disk (Fig. 12a).

Proposition 2.7 [Arn9, 20]. *The minimum of the Dirichlet functional is attained on the function u that depends only on the distance to the center of the disk and whose sets $\{(x, y) \mid u(x, y) \le c\}$ of smaller values have the same areas as those of the initial function u_0 (Fig. 12b).*

The proof essentially is the application of the isoperimetric and Schwarz inequalities. $\square$
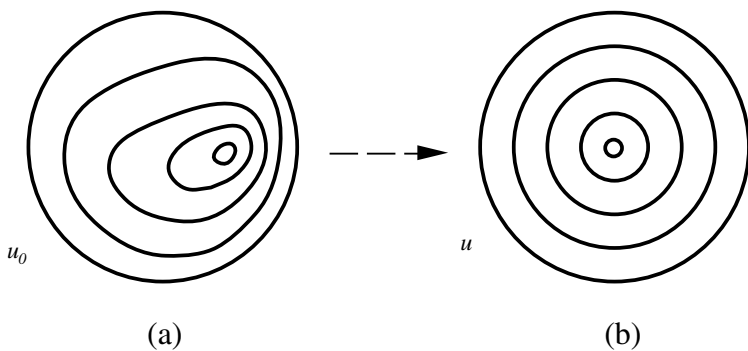
FIGURE 12.   Levels of (a) a function u_0 with the only critical point (maximum) inside the disk, and (b) the centrally symmetrical Dirichlet minimizer u among the functions that are area-preserving rearrangements of u_0.

If the initial function has several critical points (say, two maxima and a saddle point, Fig. 13), the situation is far more subtle. Numerical experiments in [Mof4, Baj] suggest various types of minimizers according to the steepness of the initial function u_0, all having "singular" lines. We refer to the extensive surveys [Mof2,4, MoT] (and references therein) for a discussion of the formation of field singularities in a fluid under the relaxation to an extremal state. The obstructions to such relaxation in three dimensions are described in Chapter III.

If instead of the initial function u_0 one prescribes just its boundary conditions, then one may obtain an infinite number of C^∞-steady solutions (or minimizers) for the problem in a rectangle, and a unique solution in the analytic category [Tro].
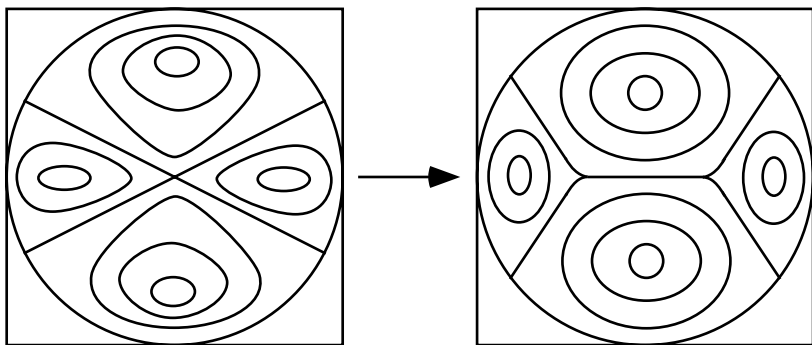
FIGURE 13.   A minimizer of the Dirichlet problem for a function with two maxima has a singular line (see [Baj]).

Theorem 2.8. *A smooth minimizer u of the Dirichlet Problem 2.1′ on a Riemannian manifold M obeys the following condition: The gradients of the functions u and Δu are collinear at every point of M.*

In other words, the extremal functions u have the "same" level curves as their Laplacians: Locally there is a function $F : \mathbb{R} \to \mathbb{R}$ such that $\Delta u = F(u)$. This is just a two-dimensional reformulation of Theorem 2.2. For instance, the axial symmetric function with its only critical point in the disk (Fig. 12) not only has the energy minimum among all diffeomorphic fields, but also has the energy maximum among all isovorticed fields [KLe].

The Dirichlet Problem 2.1′ in higher dimensions has applications to scalar dynamos [Bay2] and the theory of equilibrium of a confined plasma [LS1]. One can show that Theorem 2.8 holds in n dimensions. (Hint: adapt the proof of Theorem 2.2.)

Remark 2.9. As discussed above, minimizers of energy (i.e., of the Dirichlet integral) among all smooth area-preserving changes of coordinates in a given function correspond to steady flows. The problem of existence of smooth minimizers is still open in any reasonable generality.

This problem admits a natural extension to a more general class of functions (for instance, from the L^p or Sobolev spaces), to all measure-preserving rearrangements of such functions on measure spaces, and to general variational functionals. There is vast literature on the existence of (usually, nonsmooth) extrema of variational problems in this setting and on their relation to 2-D hydrodynamics, when one minimizes (or maximizes) the energy functional among the rearrangements (see [Bej, ATL, Bur]). In Section 2.D we discuss a different variational principle proposed in [Shn3] for two-dimensional flows, where one confines oneself to the same energy level, but constructs a partial order on functions. Minimal elements in this partial order correspond to steady flows.

A step towards the intrinsic characterization of the weak closure (in H_0^1) of the set of functions obtained from a given one by composing it with diffeomorphisms (not necessarily volume preserving) of the domain is obtained in [LS3]. It is done under the assumption that the function is *craterless*; i.e., in an appropriate weak sense it has no local minima in the interior of the domain. The authors define a subspace of this weak closure that captures robust (under weak limits) topological properties of the level sets.

2.C. Relation of two variational principles

We have observed that the smooth extremals of the energy functional among the vector fields diffeomorphic to a given one commute with their vorticities, and hence they coincide with the description of ideal steady flows (cf. Remark 1.4 and Theorem 2.2). This coincidence of the solutions in two problems is a manifestation of the duality of the two variational principles: in ideal hydrodynamics and in magnetohydrodynamics.

The steady solutions in ideal hydrodynamics correspond to critical points of the energy $\int (v, v)^2 / 2$ among all *isovorticed fields*, i.e., among the fields whose vorticities differ by the action of a volume-preserving diffeomorphism. In the Lie-algebraic language, steady flows correspond to stagnation points of the energy

functional on the *coadjoint orbits* of the group of volume-preserving diffeomorphisms $S\operatorname{Diff}(M^n)$ (see Chapter I). On the other hand, in the above problem we are looking for an energy minimizer within the class of *diffeomorphic fields*, i.e., on the *adjoint orbits* of the same group of volume-preserving diffeomorphisms. Note that the latter principle of energy minimization among the diffeomorphic fields is encountered in the MHD theory (see Chapter III, or, e.g., [Arn9, 20, Bej, Ser2, Mof2, 4]).

Theorem 2.2 above can now be reformulated as follows: "Extrema for both variational principles coincide." This statement materializes in a very general phenomenon valid for any nondegenerate quadratic form E on an arbitrary Lie algebra $\mathfrak{g}$. Let E^* be the quadratic form on the dual space $\mathfrak{g}^*$ corresponding to the form E on $\mathfrak{g}$. If the form $E(x) = \frac{1}{2}\langle x, Ax \rangle$ is defined by means of an (invertible) inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$, then E^* is determined by $E^*(y) = \frac{1}{2}\langle A^{-1}y, y \rangle$ for any $y \in \mathfrak{g}^*$.

Theorem 2.10. *Conditional extrema of the quadratic functional E on adjoint orbits in a Lie algebra $\mathfrak{g}$ are sent by the inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$ to the conditional extrema of the quadratic form E^* on the coadjoint orbits in $\mathfrak{g}^*$.*

Proof. Let x_0 be a point of the Lie algebra $\mathfrak{g}$, and $\mathcal{O}$ the adjoint orbit of the point x_0. An arbitrary vector ζ of the tangent space $T_{x_0}\mathcal{O}$ can be written by definition as a variation of x_0, i.e., as $\zeta = \operatorname{ad}_\eta x_0$ for some element $\eta \in \mathfrak{g}$. Therefore, one has the following expression for the variation of the energy functional $E(v) = \frac{1}{2}\langle x, Ax \rangle$ along the vector ζ:

$$dE(\zeta) = \langle \zeta, Ax_0 \rangle = \langle \operatorname{ad}_\eta x_0, Ax_0 \rangle = \langle x_0, \operatorname{ad}_\eta^*(Ax_0) \rangle$$
$$= \langle A^{-1}y_0, \operatorname{ad}_\eta^* y_0 \rangle = dE^*(\zeta^*),$$

where $y_0 \in \mathfrak{g}^*$ denotes the image of x_0 under the inertia operator ($y_0 = Ax_0$), and the vector $\zeta^* = \operatorname{ad}_\eta^* y_0$ represents an arbitrary vector tangent to the coadjoint orbit $\mathcal{O}^*$ of the point y_0.

Now assume that $x_0 \in \mathfrak{g}$ is a critical point of the function $E(x)$ restricted to the adjoint orbit $\mathcal{O}$ of x_0. Then the differential of E vanishes on the tangent space $T_{x_0}\mathcal{O}$ and so does the differential of E^* restricted to the tangent space to $\mathcal{O}^*$ at y_0. Hence y_0 is a critical point of E^* restricted to the coadjoint orbit $\mathcal{O}^*$. ☐

2.D. Semigroup variational principle for two-dimensional steady flows

In [Shn3], Shnirelman proposed a different variational principle in two dimensions that recovers some of the steady solutions of the Euler equation. Roughly speaking, instead of the energy minimization among all isovorticed fields, one can stay among the fields with the same energy and construct a partial order on their vorticities. In a sense, the extremal fields obtained by this method have the most mixed vorticity functions.

Consider a bounded connected two-dimensional domain $M \subset \mathbb{R}^2$ with a measure μ and boundary $\Gamma = \partial M$. We wish to describe generalized area-preserving mappings of M into itself that are not necessarily one-to-one. It is natural to define them in terms of their actions on functions on M.

Definition 2.11. A *polymorphism* is a bounded operator $\widetilde{K}$ in $L^2(M, \mathbb{R})$ of the form

$$\widetilde{K}u(x) = \int_M K(x, y)u(y)\mu_y,$$

where the (distributional) kernel $K(x, y)$ obeys the following conditions:

(i) $K(x, y) \geq 0$; i.e., $K(x, y)$ is a nonnegative measure on $M \times M$;

(ii) $\int\limits_M K(x, y)\mu_x \equiv 1$ for every $y \in M$; and

(iii) $\int\limits_M K(x, y)\mu_y \equiv 1$ for every $x \in M$.

Examples 2.12. Two obvious, yet important, examples of such operators are:

(A)   Let $\varphi \in S\,\mathrm{Diff}(M)$ be an area-preserving diffeomorphism of M. Set $K_\varphi(x, y) = \delta(y - \varphi^{-1}(x))$, where $\delta(*)$ is the 2-dimensional δ-function. Then the operator $\widetilde{K}_\varphi$ whose kernel is $K_\varphi(x, y)$ sends a function $u(x)$ to the function $u(\varphi^{-1}(x))$ and is unitary in $L^2(M)$.

(B)   If $K_0(x, y) \equiv 1/\mu(M)$ where $\mu(M)$ is the total measure of M, the operator $\widetilde{K}_0$ maps a function $u(x)$ to the constant that is the mean value of $u(x)$.

In a sense, an arbitrary operator $\widetilde{K}$ interpolates between those two extreme cases.

Conditions (ii) and (iii) generalize the volume-preserving property of diffeomorphisms: They demand that the probabilistic measure of the "image" of the element dy and the "inverse image" of the element dx under an operator $\widetilde{K}$ be equal to the measures of the elements dy and dx, respectively.

All polymorphisms form a (weakly compact) semigroup $\mathcal{P}$ of (contractive, or more precisely, nonexpanding) operators in $L^2(M)$. The operators $\widetilde{K}_\varphi$ corresponding to diffeomorphisms constitute a weakly dense subset of $\mathcal{P}$. Representations of the group of diffeomorphisms can be extended to the semigroup of polymorphisms [Ner2].

Definition 2.13. The *partial ordering* in $L^2(M)$ is dictated by the action of $\mathcal{P}$: $f \prec g$ if there exists an operator $\widetilde{K} \in \mathcal{P}$ such that $f = \widetilde{K}g$. If $f \prec g$ and $g \prec f$, we say that f and g are equivalent: $f \sim g$.

The following property of the relation $\prec$ will be useful in the sequel.

Proposition 2.14 [Shn3]**.** *If* $f, g \in L^2(M)$ *and* $f \prec g$, *then* $\|f\|_{L^2} \leq \|g\|_{L^2}$. *For* $f \prec g$ *the equality of the norms* $\|f\|_{L^2} = \|g\|_{L^2}$ *is possible if and only if* $g \prec f$.

Let $L^{2,2}(M)$ be the Sobolev space that consists of functions φ obeying

$$\sum_{|k| \leq 2} \|D^k \varphi\|^2_{L^2(M)} < \infty, \qquad \varphi\big|_{\partial M} = \text{const}.$$

Definition 2.15. Given a function $\varphi \in L^{2,2}(M)$, denote by $\bar{\Omega}_\varphi$ the set of such functions $\psi \in L^{2,2}(M)$ that

$$(2.2a) \qquad\qquad\qquad \Delta \psi \prec \Delta \varphi.$$

If φ is regarded as a stream function for a fluid flow, then the set $\bar{\Omega}_\varphi$ contains the fields isovorticed with φ, i.e., the fields with the stream functions ψ for which there exists a diffeomorphism $g : M \to M$ such that $\Delta \psi(x) = \Delta \varphi(g(x))$. These fields constitute the coadjoint orbit $\mathcal{O}_\varphi$ of φ.

Let $\Omega_\varphi \subset \bar{\Omega}_\varphi$ be the set of stream functions ψ obeying one extra condition of the conservation of energy:

$$(2.2b) \qquad\qquad\qquad E(\psi) = E(\varphi),$$

where $E(\psi) = \frac{1}{2}\|\nabla \psi\|^2_{L^2}$ is the kinetic energy of the flow with the stream function ψ.

An element $\nu \in \Omega_\varphi$ is *minimal* relative to the partial ordering on Ω_φ if $\Delta \nu' \sim \Delta \nu$ whenever $\nu' \in \Omega_\varphi$ and $\Delta \nu' \prec \Delta \nu$.

Theorem 2.16 [Shn3]. *For each function $\varphi \in L^{2,2}(M)$ there exists a minimal element $\nu \in \Omega_\varphi$ in the set Ω_φ.*

A minimal element is not necessarily unique. The proof is essentially a combination of the *Zorn lemma* (claiming that if for each linearly ordered decreasing chain of elements of a partially ordered set there is a lower bound, then there exists a minimal element in the set) with the relative weak compactness of the set of measures $\{K(x, y)\}$.

Theorem 2.17 [Shn3]. *Let u be a minimal element of Ω_φ. Then u is the stream function of a stationary flow, and moreover, there exists a single-valued monotone function F such that $\Delta u = F(u)$ almost everywhere in M.*

The equivalent statement is that if u is a minimal element of Ω_φ, then, for almost all points $x, y \in M$, the products $(u(x) - u(y))(\omega(x) - \omega(y))$, where $\omega := \Delta u$, all have the same sign. We refer to [Shn3] for the proof and all the details.

Remark 2.18. Though a classical solution of the Euler equation is a trajectory on the coadjoint orbit $\mathcal{O}_\varphi$ for some function φ, for large times the flow transformations become similar to the mixing described by polymorphisms. These are the heuristics lying behind the relation between the minimal elements and the stationary solutions of the Euler equation.

Remark 2.19. For a non-simply connected M, the boundary conditions for functions in the space $L^{2,2}(M)$ are $\varphi|_{\Gamma_i} \equiv \text{const}_i$, where Γ_i is a connected component of ∂M. In the latter case the set $\bar{\Omega}_\varphi$ consists of the functions ψ that in addition to the condition (2.2a) satisfy the property

$$(2.2c) \qquad \int\limits_{\Gamma_i} \frac{\partial \psi}{\partial n}\, ds = \int\limits_{\Gamma_i} \frac{\partial \varphi}{\partial n}\, ds \qquad \text{for all} \quad i.$$

Property (2.2c) follows from (2.2a) for a simply connected M.

One can classify minimal elements of the "orbit" Ω_φ by comparing their energy to other points of the set $\bar{\Omega}_\varphi \supset \Omega_\varphi$ consisting of the stream functions obeying conditions (2.2a) and (2.2c), but without the requirement (2.2b) on the energy.

Theorem 2.20 [Shn3]. *Each minimal element $u \in \Omega_\varphi$ is one of the following three types:*

    (a) *energy-excessive, i.e., $E(u) \geq E(\psi)$,*
    (b) *energy-deficient, i.e., $E(u) \leq E(\psi)$, or*
    (c) *neutral, i.e., $E(u) = E(\psi)$*

for all $\psi \in \bar{\Omega}_\varphi$. All the minimal elements of Ω_φ are of the same type.

Problem 2.21. It would be interesting to relate these types of minimal elements and the above variational principle to various types of energy relaxation discussed in Section 2.B (cf. numerical simulations in [Mof4, Baj]).

This variational principle might be a basis for formulating for *semigroups* an analogue of the (geodesic) variational principle for groups (Chapter I). In Section IV.7.G, we discuss a natural passage from the geodesics on the group of volume-preserving diffeomorphisms of a manifold to the extremals of the least action principle for the so-called generalized flows (which are similar to the semigroup of polymorphisms), i.e., the passage from classical fluid motions to generalized solutions of the Euler equation; see [Bre1, Shn5].

§3. Stability of stationary points on Lie algebras

In order to study the stability of stationary fluid flows in the next section, we obtain below a stability criterion for the Euler equation on an arbitrary Lie algebra.

Consider a system of ordinary differential equations

$$(3.1) \qquad \dot{x} = f(x), \quad x \in \mathbb{R}^n.$$

Definition 3.1. A point x_0 at which $f(x_0) = 0$ is *(Lyapunov) stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x(t) - x_0| < \varepsilon$ for all $t > 0$, provided that $|x(0) - x_0| < \delta$.

Assume that we are also given a foliation in the space $\mathbb{R}^n$. A point x_0 is called *regular* for the foliation if the partition of a neighborhood of x_0 into the leaves of the foliation is diffeomorphic to a partition of the Euclidean space into parallel planes (in particular, all leaves near the point x_0 have the same dimension).

Example 3.2. In the case of the Lie algebra $\mathfrak{so}(3)$ the orbits form a partition of three-dimensional space $\mathfrak{so}(3) \simeq \mathbb{R}^3$ into spheres centered at 0 and the point 0 itself. Then all points of the space $\mathbb{R}^3$, except the origin, are regular for the partition into orbits.

Suppose now that the system (3.1) leaves the foliation invariant, and E is a first integral of the system such that

   (i) x_0 is a critical point of E restricted to the leaf containing x_0;
  (ii) x_0 is a regular point of the foliation; and
 (iii) the second differential of E restricted to the leaf of x_0 is a nondegenerate quadratic form.

The following statement is essentially a reformulation of Lagrange's theorem.

Theorem 3.3. *A point x_0 obeying conditions (i)–(iii) is a stationary point of the system (3.1). If, in addition, the second differential of E restricted to the leaf of x_0 is positively or negatively defined, then the point x_0 is (Lyapunov) stable.*

Proof. If y is a coordinate on the leaf such that $y(x_0) = 0$, then the function E restricted to the leaf can be written as $E(y) = E_0 + \frac{1}{2}(E_2 y, y) + \mathcal{O}(y^3)$ as $y \to 0$, where the matrix E_2 is symmetric: $(E_2 y, z) = (y, E_2 z)$. Hence the time derivative along the trajectories of our system is

$$\dot{E} = (E_2 y, \dot{y}) + \mathcal{O}(y^2)\dot{y} \qquad \text{as} \quad y \to 0.$$

If $\dot{y} \neq 0$ at the origin $y = 0$, then one can choose a point y arbitrarily close to the origin such that $(E_2 y, \dot{y}) \neq 0$. The latter contradicts the invariance of E. Therefore, $\dot{y} = 0$, and x_0 is a stationary point.

The regularity of the leaves near x_0 implies that on every neighboring leaf there exists near x_0 a point that is a conditional maximum or minimum of E. The stability part of the statement is evident (Lagrange, Dirichlet, etc.): The definiteness of E ensures that in every leaf near x_0 the E-levels form a family of ellipsoid-like hypersurfaces. Every trajectory of the system (3.1) that begins inside such an ellipsoid will never leave it, due to the invariance of E and of the foliation (see Fig. 14).   □

Let ν be a stationary point of the Euler equation on a Lie algebra $\mathfrak{g}$ (see Chapter I). The space $\mathfrak{g}$ is foliated by the images of the coadjoint orbits in the algebra, and we suppose that ν is a regular point of the foliation.
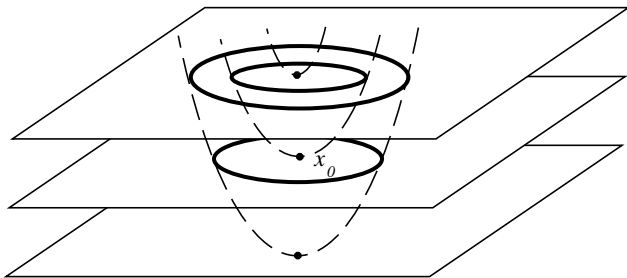
FIGURE 14.   Trajectories enclosed in ellipsoid-like intersections of foliation leaves (here, horizontal planes) and energy levels (paraboloids) will never leave a vicinity of the stationary point.

Theorem 3.4 [Arn4, 16]. *The second differential of the kinetic energy restricted to the image of an orbit of the coadjoint representation in the algebra $\mathfrak{g}$ is given at a critical point $v \in \mathfrak{g}$ by the formula*

$$(3.2) \qquad 2\delta^2 E\big|_v(\xi) = \langle B(v, f), B(v, f)\rangle + \langle [f, v], B(v, f)\rangle,$$

where ξ is a tangent vector to this image expressed in terms of $f \in \mathfrak{g}$ by the formula $\xi = B(v, f)$, and $B(\cdot, \cdot)$ is the operation on $\mathfrak{g}$ defined by (I.4.3).

Corollary 3.5. *If the quadratic form above is positive or negative definite, then the stationary point v is a stable solution of the Euler equation.*

Example 3.6. In the case of the rigid body ($\mathfrak{g} = \mathfrak{so}(3)$), the coadjoint orbits are spheres centered at zero, while the levels of the kinetic energy form a family of ellipsoids. The energy restricted to every orbit has 6 critical points (being points of tangency of the sphere with the ellipsoids): 2 maxima, 2 minima, and 2 saddles (Fig. 15). The maxima and minima correspond to the stable rotations of the rigid body about the shortest and the longest axes of the inertia ellipsoid. The saddles correspond to the unstable rotations about its middle axis.

We emphasize that the question under discussion is not stability "in a linear approximation," but the actual Lyapunov stability (i.e., with respect to finite perturbations in the nonlinear problem). The difference between these two forms of stability is substantial in this case, since our problem has a Hamiltonian character. For Hamiltonian systems asymptotic stability is impossible, so stability in a linear approximation is always neutral and inconclusive in regard to the stability of an equilibrium position of the nonlinear problem.

Remark 3.7. In general, an indefinite quadratic form $\delta^2 E$ does not imply instability of the corresponding point. An equilibrium position of a Hamiltonian system can be stable even if the Hamiltonian function at this position is neither a maximum
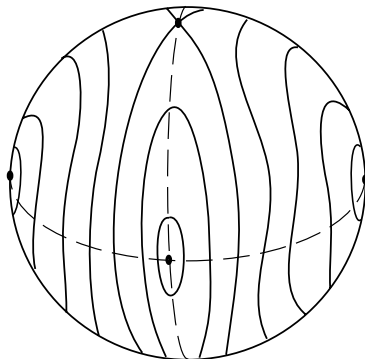
FIGURE 15.   Energy levels on a coadjoint orbit of the Lie algebra $\mathfrak{so}(3, \mathbb{R})$ of a rigid body.

nor a minimum. The quadratic Hamiltonian

$$E = \omega_1 \frac{p_1^2 + q_1^2}{2} - \omega_2 \frac{p_2^2 + q_2^2}{2}$$

is the simplest example of this kind. Note that the behavior of the corresponding eigenvalues under the introduction of a small viscosity is different: $\pm i\omega_1$ are moving into the left (stable) hyperplane, while $\pm i\omega_2$ are moving into the right (unstable) one.

Proof of Theorem 3.4. The action of an element $\varepsilon \cdot f \in \mathfrak{g}$ on a point ν is given by the Taylor expansion for motion along a coadjoint orbit; cf. formula (2.1):

$$\nu \mapsto \bar{\nu} = \nu + \varepsilon \cdot \xi + \frac{\varepsilon^2}{2} \cdot \zeta + O(\varepsilon^3), \qquad \varepsilon \to 0,$$

where $\xi = B(\nu, f)$, $\zeta = B(B(\nu, f), f)$. Substitute $\bar{\nu}$ into the expression for the energy $E(\bar{\nu}) = \frac{1}{2}\langle \bar{\nu}, \bar{\nu} \rangle$:

$$E(\bar{\nu}) = E(\nu) + \varepsilon \cdot \delta E + \varepsilon^2 \cdot \delta^2 E + O(\varepsilon^3), \qquad \varepsilon \to 0,$$

where $\delta E = \langle \nu, \xi \rangle$ and $2\delta^2 E = \langle \xi, \xi \rangle + \langle \nu, \zeta \rangle$.

The first variation of the energy vanishes at ν:

$$\delta E = \langle \nu, B(\nu, f) \rangle = -\langle B(\nu, \nu), f \rangle = 0,$$

since ν is stationary, and therefore $B(\nu, \nu) = 0$.

The required expression (3.2) for $\delta^2 E$ follows due to the identity

$$\langle \nu, B(B(\nu, f), f) \rangle = \langle [f, \nu], B(\nu, f) \rangle.$$

Now we would like to show that the quadratic form $\delta^2 E$ depends on $\xi = B(\nu, f)$ rather than on f, so it is indeed a form on the tangent space in $\mathfrak{g}$.

First verify that the auxiliary bilinear form $C(x, y) := \langle [x, \nu], B(\nu, y) \rangle$ is symmetric: $C(x, y) = C(y, x)$. It readily follows from the definition of B, the Jacobi

identity in $\mathfrak{g}$, and from the stationarity condition $B(v, v) = 0$ that

$$\langle [x, v], B(v, y) \rangle = \langle B(v, [v, x]), \ y \rangle = \langle [[v, x], y], \ v \rangle$$
$$= \langle [v, [x, y]], \ v \rangle + \langle [x, [y, v]], \ v \rangle$$
$$= \langle B(v, v), \ [x, y] \rangle + \langle B(v, x), \ [y, v] \rangle = \langle [y, v], B(v, x) \rangle.$$

Finally, assume that $B(v, f_1) = B(v, f_2)$ and show that the corresponding values of $\delta^2 E$ coincide. Set $x = f_1 - f_2, y = f_1$, and notice that $B(v, x) = 0$. The expression (3.2) for $\delta^2 E$, combined with the symmetry of $C(x, y)$, gives the desired identity:

$$2(\delta^2 E(f_1) - \delta^2 E(f_2)) = \langle [x, v], B(v, y) \rangle = \langle [y, v], B(v, x) \rangle = 0.$$

Thus, the quadratic form $\delta^2 E$ indeed depends on $\xi = B(v, f)$, and Theorem 3.4 is proved. $\qquad\square$

Remark 3.8. For the Euler equation on a Lie algebra $\mathfrak{g}$ consider the *equation in variations* at a stationary point v:

$$(3.3) \qquad\qquad\qquad \dot{\xi} = B(v, \xi) + B(\xi, v).$$

Proposition 3.9. *The quadratic form $d^2 E$ is the first integral of the equation in variations (3.3).*

Proof. The proposition can be verified by the following straightforward calculation. From (3.2), it follows that

$$\frac{d}{dt}\delta^2 E = \langle \xi, \dot{\xi} \rangle + \langle [f, v], \dot{\xi} \rangle.$$

Therefore the substitution of $\dot{\xi}$ from the equation in variations (3.3) leads to

$$\frac{d}{dt}\delta^2 E = \langle \xi, B(v, \xi) \rangle + \langle \xi, B(\xi, v) \rangle + \langle [f, v], B(v, \xi) \rangle + \langle [f, v], B(\xi, v) \rangle$$

$$= \langle \xi, B(\xi, v) \rangle + \langle [\xi, v], \xi \rangle + \langle [v, [f, v]], \xi \rangle$$

$$= \langle [v, [f, v]], B(v, f) \rangle = -\langle [f, v], B(v, [f, v]) \rangle = 0.$$

$$\square$$

§4. Stability of planar fluid flows

The analogy between the equations of a rigid body and of an incompressible fluid enables one to study stability of steady flows by considering critical points of the energy function on the sets of isovorticed vector fields (i.e., on the coadjoint orbits of the diffeomorphism group).

This approach was initiated in [Arn4], and we refer to Fjørtoft [Fj] as a predecessor, and to [HMRW] for further applications manifesting the fruitfulness of

this method for a variety of dynamical systems. In this section we touch on a few selected facts.

In Section 3 we saw that the variational approach to the study of the stationary solutions of the Euler equation of an incompressible fluid suggests that:

(i) A steady fluid flow is distinguished from all flows isovorticed to it by the fact that it is a (conditional) critical point of the kinetic energy.

(ii) If the indicated critical point is actually an extremum, i.e., a local conditional maximum or minimum, and this extremum is nondegenerate (the second differential d^2E is positive or negative definite), then (under some regularity condition) the stationary flow is Lyapunov stable.

Though these assertions do not formally follow from the theorems of Section 3 because of the infinite-dimensionality of our consideration here, one can justify the final conclusion about stability without justifying the intermediate constructions.

4.A. Stability criteria for steady flows

Let M be a two-dimensional domain, say, an annulus with a steady flow in it (Fig. 16). In what follows we show, in particular, that the steady flow in M is stable if its stream function ψ satisfies the following condition on the velocity profile:

$$(4.1) \qquad 0 < c \le \frac{\nabla \psi}{\nabla \Delta \psi} \le C < \infty$$

for some constants c and C.

For an arbitrary stationary flow in two dimensions the gradient vectors of the stream function and of its Laplacian are collinear. Therefore the ratio $\nabla \psi / \nabla \Delta \psi$ makes sense. Furthermore, in a neighborhood of every point that is not critical for the vorticity function $\Delta \psi$, the stream function ψ is a function of the vorticity.

We begin the study of the two-dimensional case by obtaining the following explicit expression for the second variation of the energy.
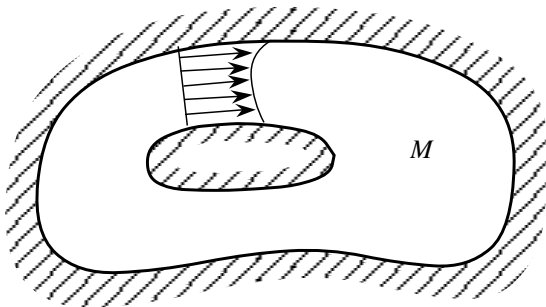
FIGURE 16.   A profile of a stable steady flow in an annulus.

Theorem 4.1 [Arn6, 16]. *The second variation of the energy E on the set of fields isovorticed to a given steady field v with the stream function ψ is*

$$\delta^2 E\big|_v = \frac{1}{2} \iint_M \left((\delta v)^2 + \frac{\nabla\psi}{\nabla\Delta\psi}(\delta\omega)^2 \right) dxdy,$$

where δv is a variation of the velocity field, $\delta\omega$ is the corresponding variation of the vorticity function $\omega = \operatorname{curl} v = \Delta\psi$, and $dxdy$ is the area form in M.

Remark 4.2. The condition (4.1) on the ratio $\nabla\psi/\nabla\Delta\psi$ implies that the quadratic form $\delta^2 E$, with respect to δv, is positively defined.

In the case of the negative ratio $\nabla\psi/\nabla\Delta\psi$ satisfying

$$0 < c \le -\frac{\nabla\psi}{\nabla\Delta\psi} \le C < \infty,$$

the form $\delta^2 E$ is negatively defined, provided that the inequality $\|\nabla\varphi\|_{L^2}^2 \le \alpha\|\Delta\varphi\|_{L^2}^2$ holds for all $\varphi \in C^2(M)$ with $0 < \alpha < c$. The latter inequality is essentially an estimate on the first eigenvalue of the Laplace operator in the domain M, and it relies on the shape and size of the domain.

Proof. Formula (3.2) for the second variation of the energy $E = \frac{1}{2}\iint_M (v,v)dxdy$ gives

$$(4.2) \qquad 2\delta^2 E = \iint_M \left((\delta v)^2 + (\delta v, [f, v]) \right) dxdy,$$

where $\delta v = B(v, f)$.

Integrating by parts the second term, we come to

(4.3)
$$\iint_M (\delta v, [f, v])\, dxdy = \iint_M (\delta v, \operatorname{curl}(f \times v))\, dxdy = \iint_M (\delta\omega)\cdot(f \times v)\, dxdy$$

with evident notations: $f \times v$ is a function on M whose value at any point is the oriented area of the parallelogram spanned by f and v, and $\operatorname{curl}(f \times v) = \operatorname{sgrad}(f \times v)$. The formula $v = \operatorname{sgrad}\psi = (-\psi_y, \psi_x)$ implies that

$$f \times v = f \times (\operatorname{sgrad}\psi) = (f, \nabla\psi).$$

On the other hand, for $\omega = \Delta\psi$, the variation $\delta\omega$ is the derivative of ω along the field f:

$$\delta\omega = L_f\omega = (f, \nabla\Delta\psi).$$

The comparison of the two formulas above immediately gives

$$f \times v = \frac{\nabla\psi}{\nabla\Delta\psi}\delta\omega,$$

which, along with (4.2–4.3), implies the statement of the theorem.   □

The above heuristic consideration of stability, based on the definiteness of the quadratic differential of the kinetic energy $\delta^2 E$, can be justified to obtain the actual stability with the following *a priori* bound.

Theorem 4.3 (Stability Theorem, [Arn6, 16]**).** *Suppose that the stream function of a stationary flow, $\psi = \psi(x, y)$, in a region M is a function of the vorticity function (i.e., of the function $\Delta\psi$) not only locally but globally. Suppose that the derivative of the stream function with respect to the vorticity satisfies the inequality*

$$c \le \frac{\nabla\psi}{\nabla\Delta\psi} \le C, \qquad where \quad 0 < c \le C < \infty.$$

Let $\psi + \varphi(x, y, t)$ be the stream function of another flow, not necessarily station-ary. Assume that at the initial moment, the circulation of the velocity field of the perturbed flow (with the stream function $\psi + \varphi$) around every boundary com-ponent of the region M is equal to the circulation of the original flow (with the stream function ψ). Then the perturbation $\varphi = \varphi(x, y, t)$ at every moment of time is bounded in terms of the initial perturbation $\varphi_0 = \varphi(x, y, 0)$ by the inequality

$$\iint_M (\nabla\varphi)^2 + c(\Delta\varphi)^2 dxdy \le \iint_M (\nabla\varphi_0)^2 + C(\Delta\varphi_0)^2 dxdy.$$

Theorem 4.3′ (Second Stability Theorem, [Arn6, 16]**).** *If the stationary flow satisfies the condition*

$$c \le -\frac{\nabla\psi}{\nabla\Delta\psi} \le C \qquad with \quad 0 < c \le C < \infty$$

(as well as other assumptions of the preceding theorem), then the perturbation φ is bounded in terms of φ_0 by the inequality

$$(4.4) \qquad \iint_M c(\Delta\varphi)^2 - (\nabla\varphi)^2 dxdy \le \iint_M C(\Delta\varphi_0)^2 - (\nabla\varphi_0)^2 dxdy.$$

Remark 4.4. If for a certain α satisfying $0 < \alpha < c$ the inequality $\|\nabla\varphi\|^2_{L^2} \le \alpha\|\Delta\varphi\|^2_{L^2}$ holds for all $\varphi \in C^2(M)$, then the quadratic form $\iint_M c(\Delta\varphi)^2 - (\nabla\varphi)^2 dxdy$ is positive definite:

$$\iint_M c(\Delta\varphi)^2 - (\nabla\varphi)^2 dxdy \ge (c - \alpha) \iint_M (\Delta\varphi)^2 dxdy.$$

Therefore it follows from (4.4) that

$$\iint_M (\Delta\varphi)^2 dxdy \le \frac{C}{c - \alpha} \iint_M (\Delta\varphi_0)^2 dxdy,$$

which manifests the stability of the stationary flow ψ.

The underlying heuristic idea of the proof of the Stability Theorem is as fol-lows. A first integral $H(\varphi)$ having a nondegenerate minimum or maximum at the

stationary point ψ can be regarded as a squared "norm" (setting $H(\psi) = 0$). It gives us control of the trajectory φ_t in the norm that is positive in a punctured neighborhood of ψ on the set of isovorticed fields.

Example 4.5. Consider a circular motion with the stream function $\psi = \psi(\rho)$, $\rho = \sqrt{x^2 + y^2}$, in the annulus $M = \{R_1 \leq \rho \leq R_2\}$. Rewriting the Laplace operator in polar coordinates, we get the following sufficient condition for stability: If the ratio $\psi'/(\psi'' + \frac{1}{\rho}\psi')'$ does not change sign, then the flow is stable (see [Arn16]).

Example 4.6. Consider a planar shear flow in the strip $0 \leq y \leq 2\pi$ in the (x, y)-plane with a velocity profile $v(y)$ (i.e., with a velocity field $(v(y), 0)$, Fig. 17). Such a flow is stationary for every velocity profile.

The form $\delta^2 E$ is positively or negatively defined if the velocity profile has no zeros and no points of inflection (i.e., $v \neq 0$ and $v_{yy} \neq 0$). The conclusion, that the planar parallel flows are stable, provided that there are no inflection points in the velocity profile, is a nonlinear analogue of the so-called Rayleigh theorem. Profiles with the ratio $v/v_{yy} > 0$ and $v/v_{yy} < 0$ are sketched in Figs. 17a and 17b, respectively.

(a)   (b)
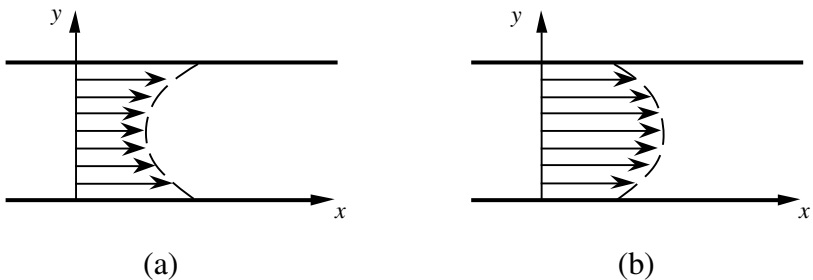
FIGURE 17.   Lyapunov stable fluid flows in a strip. Profiles with the ratio (a) $v/v_{yy} > 0$ and (b) $v/v_{yy} < 0$.

To make the region of the flow compact, we impose the periodicity condition $x \pmod X$ along the x-coordinate and obtain the torus $\{(x, y) \mid x \pmod X, y \pmod{2\pi}\}$. Fix the velocity field $v = (\sin y, 0)$ determined by the stream function $\psi = -\cos y$. Its vorticity is $\omega = -\cos y$. The velocity profile has two inflection points, but the stream function can be expressed as a function of the vorticity. The ratio $\nabla\psi/\nabla\Delta\psi$ is equal to minus one. By applying the Second Stability Theorem, we have obtained the stability of our stationary flow in the case when

$$\int_0^{2\pi} \int_0^X (\Delta\varphi)^2 dxdy \geq \int_0^{2\pi} \int_0^X (\nabla\varphi)^2 dxdy$$

for all functions φ of period X in x and 2π in y. It is easy to calculate that the last inequality is satisfied for $X \leq 2\pi$ and is violated for $X > 2\pi$.

Thus the Second Stability Theorem implies the stability of a sinusoidal stationary flow on a short torus when the period in the direction of the basic flow (X) is less than the width of the flow (2π). On the other hand, one can directly verify that on a long torus (for $X > 2\pi$) our sinusoidal flow is unstable [MSi]. Hence, in this example, the sufficient condition for stability from the Second Stability Theorem turns out to be necessary as well.

Stability of certain plane-parallel and spherical two-dimensional flows was considered in [Dik].

Proof of Stability Theorem. Assume that the stream function ψ and the vorticity function $\omega = \Delta\psi$ are related by means of $\psi = \Psi(\Delta\psi)$, and set $\Phi(\tau) := \int^\tau \Psi(\theta)\,d\theta$ to be the primitive of $\Psi(\theta)$. Then the second derivative Φ'' evaluated at the function $\Delta\psi$ is $\Phi''(\Delta\psi) = \nabla\psi/\nabla\Delta\psi$, and hence for τ within the limits $\min\Delta\psi \leq \tau \leq \max\Delta\psi$, we have

$$(4.5) \qquad\qquad c \leq \Phi''(\tau) \leq C.$$

We extend the definition of $\Phi(\tau)$ to cover the whole τ-axis subject to this inequality, and in what follows Φ denotes the function extended in this way.

Form the functional

$$H_2(\varphi) = \iint_M \left(\frac{(\nabla\varphi)^2}{2} + [\Phi(\Delta\psi + \Delta\varphi) - \Phi(\Delta\psi) - \Phi'(\Delta\psi)\Delta\varphi] \right) dx\,dy.$$

Lemma 4.7. *The functional H_2 is the first integral of the Euler equation,*

$$H_2(\varphi(x, y, t)) \equiv H_2(\varphi(x, y, 0)),$$

for the stream function $\varphi(x, y, t)$ of any velocity field evolving according to the Euler equation.

Proof of Lemma. Consider the functional

$$H(u) = \iint_M \left(\frac{(\nabla u)^2}{2} + \Phi(\Delta u) \right) dx\,dy.$$

It is preserved along every solution of the Euler equation by virtue of the laws of energy and vortex conservation. Therefore, $\hat{H}(\varphi) := H(\psi + \varphi) - H(\psi)$ is also a conserved functional for a given steady flow ψ:

$$(4.6) \qquad\qquad \hat{H}(\varphi(x, y, t)) \equiv \hat{H}(\varphi(x, y, 0)).$$

Decompose $\hat{H}(\phi)$ into the sum $\hat{H}(\varphi) = H_1(\varphi) + H_2(\varphi)$, where

$$H_1(\varphi) = \iint_M \left((\nabla\varphi, \nabla\psi) + \Phi'(\Delta\psi)\Delta\varphi \right) \, dxdy,$$

$$H_2(\varphi) = \iint_M \left(\frac{(\nabla\varphi)^2}{2} + [\Phi(\Delta\psi + \Delta\varphi) - \Phi(\Delta\psi) - \Phi'(\Delta\psi)\Delta\varphi] \right) \, dxdy.$$

The term $H_1(\varphi)$ vanishes, since it is the first variation of the invariant functional $H(u)$ at the stationary flow ψ. Explicitly, after integration by parts we have

$$H_1(\varphi) = \iint_M (-\psi\Delta\varphi + \Phi'(\Delta\psi)\Delta\varphi) \, dxdy + \oint_{\partial M} \psi \frac{\partial\varphi}{\partial n} \, d\ell.$$

Recall that $\Phi' = \Psi$ and $\Psi(\Delta\psi) = \psi$. Furthermore, by assumption the stream function ψ is constant on the boundary components Γ_i ($\partial M = \bigcup_i^n \Gamma_i$), and the perturbed fields have the same circulation around every boundary component: $\oint_{\Gamma_i} \partial\varphi/\partial n \, d\ell = 0$. Hence $H_1(\varphi) \equiv 0$. Therefore $\hat{H}(\varphi) = H_2(\varphi)$, and in accordance with (4.6), the functional $H_2(\varphi)$ is preserved. This proves Lemma 4.7.
□

Returning to the proof of the theorem, we note that it follows from (4.5) that for any h,

$$c\frac{h^2}{2} \le \Phi(\tau + h) - \Phi(\tau) - \Phi'(\tau)h \le C\frac{h^2}{2}.$$

Hence,

$$H_2(\varphi(t)) \ge \iint_M \left(\frac{(\nabla\varphi)^2}{2} + c\frac{(\Delta\varphi)^2}{2} \right) \, dxdy,$$

$$H_2(\varphi(0)) \le \iint_M \left(\frac{(\nabla\varphi_0)^2}{2} + C\frac{(\Delta\varphi_0)^2}{2} \right) \, dxdy.$$

By combining these inequalities with the invariance of $H_2(\varphi)$ we complete the proof of the Stability Theorem.
□

We leave to the reader to complete the proof of stability for the negative ratio (the Second Stability Theorem)

$$c \le -\frac{\nabla\psi}{\nabla\Delta\psi} \le C, \qquad 0 < c \le C < \infty.$$

Remark 4.8 [M-P]. Notice that the condition $0 < c \le \frac{\nabla\psi}{\nabla\Delta\psi} \le C$ cannot be obeyed in domains without boundary. Indeed, the existence of a function Ψ obeying the condition $0 < c \le \Psi'(\tau) \le C$ and such that $\psi = \Psi(\Delta\psi)$ implies the existence of the inverse function F for which $\Delta\psi = F(\psi)$, and moreover, $0 < c' \le F'(\psi) \le C'$.

On the other hand, from $\Delta\psi = F(\psi)$ one gets $\partial_{x_1}\Delta\psi = F'(\psi)\partial_{x_1}\psi$, and therefore

$$\iint_M \partial_{x_1}\psi\,(\Delta\partial_{x_1}\psi)\,dxdy = \iint_M F'(\psi)(\partial_{x_1}\psi)^2\,dxdy.$$

Integrating by parts we come to the following:

$$\int_{\partial M} \partial_{x_1}\psi\,\frac{\partial(\partial_{x_1}\psi)}{\partial n}\,d\ell - \iint_M (\nabla\partial_{x_1}\psi)^2\,dxdy = \iint_M F'(\psi)(\partial_{x_1}\psi)^2\,dxdy.$$

Now one can see that the absence of the boundary term leads to a contradiction: The left- and the right-hand sides of the equality are of different signs unless ψ is constant (the trivial case of $\partial_{x_1}\psi \equiv 0$ is treated by replacing ∂_{x_1} with ∂_{x_2}). In particular, it excludes unbounded domains (such as $M = \mathbb{R}^2$, important for meteorological and oceanographic simulations) from the scope of applicability of the Stability Theorem. A way to overcome this difficulty is to exploit the symmetry properties of the domains accompanied by the stability analysis outlined above.

Theorem 4.9 [M-P]. *In the hypotheses of the Stability Theorem, the stability result is achieved if the condition $c \leq \frac{\nabla\psi}{\nabla\Delta\psi} \leq C$ holds with $c \geq 0$.*

The proof is based on the use of a family of Lyapunov functions $H^\epsilon(\varphi)$ for which the first variation at the stationary flow ψ is given by $H_1^\epsilon(\varphi) = \epsilon\iint(\nabla\varphi, \nabla\Delta\psi)\,dxdy$.

Remark 4.10. It turns out that the stability test based on the second variation of steady flows is inconclusive in dimensions greater than two: The second variation of the kinetic energy is never sign definite in that case (see Section 5.G).

Invariants of isovorticed fields (i.e., Casimir functions of the group of area-preserving diffeomorphisms) play the role of Lagrange multipliers in the above study of the conditional extremum. We refer to the survey [HMRW] for a study of stability by combining the energy function with Casimir functions for a number of physically interesting infinite-dimensional systems. Various modifications and extensions of the *Routh* (or, *energy–Casimir*) method outlined above can be found in, e.g., [MaR, MaS, Vla1, 2, W-G].

Remark 4.11 (J. Marsden). Abbreviated guide to the energy–momentum method. For a more complete guide to the literature, see http://www.cds.caltech.edu/~marsden/

The energy–momentum (em) method extends the Arnold (or the energy–Casimir) method, which was developed for Lie–Poisson systems on duals of Lie algebras, especially those of fluid dynamical type. The motivation for this extension is threefold. First, it can deal with Lie–Poisson systems for which there are not sufficient Casimir functions available, such as 3-D ideal flow and certain problems in elasticity. In fact, [A-H] use (with hindsight) the em-method to show that 3-D equilibria for ideal flow are always formally unstable due to vortex stretching.

Other fluid and plasma situations, such as *ABC* flows and certain multiple hump situations in plasma dynamics, provided additional motivation in the Lie–Poisson setting. Second, it extends the method to systems that need not be Lie–Poisson. Examples such as rigid bodies with vibrating antennas (see [KrM]) motivate this need. Finally, it gives sharper stability conclusions in material representation (stability is modulo a *subgroup* of the symmetry group) as well as giving links with geometric phases (Berry phases); see [Pat, MMR]. This is seen already in rigid body problems.

The setting of the energy–momentum method is that of a mechanical system with symmetry with a configuration space Q and phase space T^*Q and a symmetry group G acting, with a standard momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the Lie algebra of G. One gets the Lie–Poisson case when $Q = G$.

The rough idea is first to formulate the problem on the unreduced space T^*Q. Here, relative equilibria associated with a Lie algebra element ξ are critical points of the *augmented Hamiltonian* $H_\xi := H - \langle \mathbf{J}, \xi \rangle$. One now computes the second variation $\delta^2 H_\xi(z_e)$ at a relative equilibrium z_e with the momentum value μ_e subject to the constraint $\mathbf{J} = \mu_e$ and on a space transverse to the action of G_{μ_e}. Although the augmented Hamiltonian H_ξ plays the role of $E + Casimir$ in the Arnold method, Casimir functions are not explicitly needed.

In explicit splittings based on the mechanical connection, the second variation $\delta^2 H_\xi(z_e)$ is block diagonal. In the *same* coordinates the symplectic structure has a simple block structure, so the linearized equations also have a canonical form. Even in the Lie–Poisson setting, this often leads to simpler second variations. This block diagonal structure is what gives the method its computational power. The theory for the em-method can be found in [MaS, SPM, SLM] (see also the exposition in [Mar]). For Lagrangian versions, see [Lew]. There is also a converse, building on classical work of Thompson and Tait, Chetayev, and others, which states that when one has a saddle point for $\delta^2 H_\xi(z_e)$, the addition of dissipation linearly (and hence nonlinearly) destabilizes the relative equilibrium; see [BKMR].

The energy–momentum method is effective in many examples. For instance, [LeS] dealt with the stability problem for pseudo-rigid bodies, which was thought to be analytically intractable. For the heavy top, see [LRSM]; for underwater vehicle dynamics, see [LMa]; and for *ABC* flows, see [CMa]. The em-method has also been used in the context of free boundary and Hamiltonian bifurcation problems [LMMR, LMR]. Finally, the method also extends to nonholonomic systems (systems with rolling constraints), as shown in [ZBM].

4.B. *Wandering solutions of the Euler equation*

Poincaré's recurrence theorem claims that for any volume-preserving continuous mapping of a bounded region into itself, almost every moving point returns repeatedly to the vicinity of its initial position.

In particular, the phase flow of the Euler equation on any *finite-dimensional* Lie algebra acquires this property. Indeed, level surfaces of the kinetic energy (i.e., of a positively definite quadratic form) E are compact. Every trajectory of the Euler

equation belongs to the intersection of some energy level with a certain coadjoint orbit of the Lie algebra.

Proposition 4.12. *The intersections of the coadjoint orbits with the noncritical energy levels can be equipped with a natural volume form conserved by the Euler equation.*

Proof. If ω is the symplectic structure on a $2m$-dimensional coadjoint orbit $\mathcal{O}$, then the symplectic volume form $\mu = \omega^m$ is preserved by any Hamiltonian flow on the orbit. The flow with the Hamiltonian function E preserves the differential $(2m-1)$-form $\mu_E := \omega^m/dE$ on the intersections of the orbit $\mathcal{O}$ with the E-levels. These intersections are compact, due to the positive-definiteness of the form E.
$\square$

Corollary 4.13. *The Poincaré recurrence theorem is applicable in this case: almost every trajectory of the Euler equation returns at times to a neighborhood of the initial point.*

Remark 4.14. The Euler equation with a nondegenerate inertia operator has an invariant C^1-measure on the whole dual Lie algebra $\mathfrak{g}^*$ (not only on the coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$ of the group) if and only if the group G is *unimodular*, i.e., the operators ad_η are traceless for all $\eta \in \mathfrak{g}$ [Ko2].

However, the Euler equation of an ideal fluid does not enjoy the recurrence property: The passage to the infinite-dimensional case is not harmless (see [Shn6] for other peculiar features of 2-D fluid dynamics). Fix, for instance, the region $M = \{1 \leq |x| \leq 2 \mid x \in \mathbb{R}^2\}$ and consider the space $\mathcal{V}$ of C^1-smooth divergence-free vector fields in M tangent to the boundary $\partial M = \Gamma_1 \cup \Gamma_2$; Fig. 18.

Theorem 4.15 [Nad]. *There exists a smooth divergence-free vector field ξ on M (tangent to the boundary ∂M) such that for any initial condition C^1-close to ξ the corresponding solution of the Euler equation in M does not return to a vicinity of the point ξ after a certain moment of time (i.e., there exist $\varepsilon, T > 0$ such that for any initial condition $v(0) \in \mathcal{V}$ satisfying $\|v(0) - \xi\|_{C^1} < \varepsilon$, the corresponding solution $v(t)$ satisfies the inequality $\|v(t) - \xi\|_{C^1} > \varepsilon$, whereas $t > T$).*

Proof. Consider the steady flow v^* with the stream function $\psi(x) = \ln |x|$: $v^* = \mathrm{sgrad}(\ln |x|)$. Let $v^* + h$ be a C^1-small (divergence-free) perturbation of the field v^* : $\|h\|_{C^1} < \delta$.

Lemma 4.16. *There exists $\delta > 0$ such that for any perturbation h with $\|h\|_{C^1} < \delta$, the solution $v(t)$ with the initial condition $v(0) = v^* + h$ obeys the inequality $\|v(t) - v^*\|_{C^0} < \frac{1}{4}$ for all $t \geq 0$.*

Proof of Lemma. The vorticity function curl $v(t)$ of the solution $v(t)$ is transported by the flow, and so is the function $\mathrm{curl}(v(t) - v^*) = \mathrm{curl}\, v(t)$, since $\mathrm{curl}\, v^* \equiv 0$. Therefore, the C^0-norm of the function $\mathrm{curl}(v(t) - v^*)$ is conserved as well as the circulation of the field $v(t) - v^*$ along the circumferences Γ_1 and Γ_2. Therefore, the statement of the lemma is essentially the maximum principle for the stream function $\psi(t)$ of the field $v(t)$, which obeys the equation $\Delta\psi(t) = -\mathrm{curl}(v(t) - v^*)$. □
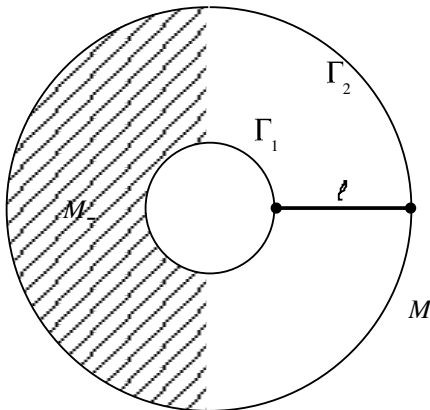
FIGURE 18.   Pick a smooth field on the annulus M vanishing on the left semiannulus M_- and whose vorticity is greater than $\delta/4$ on the segment ℓ.

Denote by $M_- = \{x \in M, x_1 < 0\}$ and $\ell = \{x \in M, x_2 = 0, x_1 > 0\}$ the semiannulus and the segment, respectively (Fig. 18). Choose some smooth divergence-free field u satisfying the following conditions:

$$\|u\|_{C^1} < \frac{\delta}{2}, \qquad u\big|_{M_-} \equiv 0, \qquad \mathrm{curl}\, u\big|_{\ell} > \frac{\delta}{4}.$$

Finally, set $\xi = v^* + u$, and notice that $\mathrm{curl}\, \xi\big|_{M_-} \equiv 0$.

Now let $v(0) \in \mathcal{V}$ be the initial condition close enough to ξ : $\|v(0) - \xi\|_{C^1} < \varepsilon$, and $v(t)$ the corresponding solution of the Euler equation on M. Such a solution defines for each $t \in \mathbb{R}$ an area-preserving diffeomorphism g^t of the annulus M. The circumferences Γ_1 and Γ_2 are mapped by g^t into themselves.

Moreover, by choosing ε to be $\varepsilon = \delta/4$, we ensure that the solution $v(t)$ is close enough to v^*. According to the lemma, the linear velocity of every point on the inner circumference Γ_1 is greater than $3/4$, while that on the outer circumference Γ_2 is smaller than $3/4$. The corresponding angular velocities are greater than $3/4$ on Γ_1 and smaller than $3/8$ on Γ_2, respectively.

The image $\ell_t := g^t(\ell)$ of the segment ℓ under the action of transformation g^t joins the points on different circumferences. The angular coordinates of the connected points diverge from each other at the rate $3t/8$. It follows that for $t > 8\pi/3$, the curve ℓ_t definitely hits $M_- : \ell_t \cap M_- \neq \emptyset$. On the other hand,

curl $v(t)$ is carried over by the flow g^t and is greater than $\delta/4 = \varepsilon$ when restricted to ℓ_t. Hence, for $t > 8\pi/3$, we have $\|\xi - v(t)\|_{C^1} > \varepsilon$. $\square$

§5. Linear and exponential stretching of particles and rapidly oscillating perturbations

In this section we study the short-wave asymptotics of the perturbations of a stationary motion of an ideal fluid (following [Arn8]).

5.A. *The linearized and shortened Euler equations*

Definitions 5.1. The 3-D Euler equation in the *vortex* (or *Helmholtz*) form

$$\frac{\partial w}{\partial t} = [v, w], \qquad \text{where} \quad w = \text{curl } v,$$

can be *linearized* in a neighborhood of a steady flow v:

(5.1)
$$\frac{\partial s}{\partial t} = [v, s] + [\text{curl}^{-1} s, w].$$

Here $[\,,\,] = -\{\,,\,\}$ is the Lie bracket (i.e., minus the Poisson bracket) of two vector fields, and s is a perturbation of the vorticity field: $\text{curl}(v + u) = w + s$, where u is a small perturbation of the steady flow v. The operator curl^{-1} is understood as the reconstruction of the divergence-free vector field from its vorticity (and from the circulations over the boundary components if $\partial M \neq \emptyset$).

   We will examine the behavior of solutions of this equation linear in s. Note that the first term on the right-hand side of (5.1) is a more powerful linear operator on functions s than the second. This means that the value of $[v, s]$ on the rapidly oscillating s of the type $s = e^{ikx}$ will contain a higher degree of the wave number k than those occurring in $[\text{curl}^{-1} s, w]$. Hence, for the rapidly oscillating perturbing field s, the second term in (5.1) may be considered as a perturbation of the first. In this way we obtain the *shortened equation*

(5.2)
$$\frac{\partial s}{\partial t} = [v, s].$$

If the stationary flow is potential ($w = 0$), the second term in equation (5.1) vanishes, and in that case the shortened equation (5.2) is the same as the linearized Euler equation (5.1). In accordance with perturbation theory [Fad], it is reasonable to assume that the shortened equation defines the continuous part of the spectrum of the linearized equation (5.1).

   The shortened equation (5.2) implies that vector s is carried by the steady flow. If the geometry of the steady flow v is known, this equation can be solved explicitly. Let g^t be a one-parameter group of diffeomorphisms generated by the field v. Then

the solution of the shortened equation is expressed in terms of its initial conditions by the formula

$$(5.3) \qquad\qquad s(t, x) = g_*^t s(0, g^{-t}(x)),$$

where g_*^t is the derivative of the image of g^t.

5.B. The action–angle variables

Below we present two lines of reasoning for the following statement.

Proposition 5.2. *For a non-Beltrami steady field (i.e., for a steady field that is not collinear with its vorticity in any region) on a closed three-dimensional manifold M, almost all solutions of the shortened equation are linearly unstable.*

Proof. If the fields v and w are not identically collinear in any region, then the manifold without boundary splits into cells in each of which the stream and vorticity lines lie on two-dimensional tori (see Theorems 1.2 and 1.10 in Section 1, or [Arn3, 4]). One can introduce the angular coordinates $\varphi = (\varphi_1, \varphi_2) \bmod 2\pi$ along the tori and the "action variable" z, which provides the numbering for the tori, such that the volume element is defined by $d\varphi_1 d\varphi_2 dz$, and the fields v and w are given by

$$v(\varphi, z) = v_1(z)\frac{\partial}{\partial \varphi_1} + v_2(z)\frac{\partial}{\partial \varphi_2}, \quad w(\varphi, z) = w_1(z)\frac{\partial}{\partial \varphi_1} + w_2(z)\frac{\partial}{\partial \varphi_2}.$$

These equations are integrable in the system of coordinates $(\varphi_1, \varphi_2, z)$. For the components of the field

$$s(t; \varphi, z) = s_1\frac{\partial}{\partial \varphi_1} + s_2\frac{\partial}{\partial \varphi_2} + s_3\frac{\partial}{\partial z},$$

by using (5.3) we obtain the expressions

$$(5.4) \qquad s_k(t; \varphi, z) = s_k(0; \varphi_0, z) + t \cdot v_k' s_3(0; \varphi_0, z), \qquad k = 1, 2,$$

$$s_3(t; \varphi, z) = s_3(0; \varphi_0, z),$$

where $\varphi_0 = \varphi - vt$, and the prime denotes the derivative with respect to z. Formulas (5.4) imply that solutions of the shortened equation (5.2) (for $v' \neq 0$) usually increase linearly with time. $\square$

Hence the conventional (exponential) instability of the linearized Euler equation for non-Beltrami flows can be due only to the second term in formula (5.1). In accordance with perturbation theory, it is reasonable to expect the appearance of a finite number of unstable discrete eigenvalues. The question of retention of the (detected above) slow instability, when passing from the shortened equation (5.2) to the complete linearized equation (5.1), is discussed in Section 5.D below.

The other possibility of exponential instability is related to the collinearity of v and w, when the action–angle variables cannot be introduced and the geometry

of the steady flow differs from the one described above (cf. [Hen]). This form of instability is examined in Section 5.E.

Remark 5.3. An integrable (non-Beltrami) steady flow can be thought of as a Hamiltonian system with two degrees of freedom that is restricted to a three-dimensional energy level. The KAM theory for volume-preserving flows on three-dimensional manifolds guarantees that under certain nondegeneracy conditions, all flows sufficiently close to the integrable ones preserve a large set of two-dimensional invariant tori (see, e.g., the survey on the KAM theory of Hamiltonian systems [AKN] or the volume-preserving case in [C-S, D-L, B-L]).

The above implies that for nonstationary Euler solutions that get close enough to a steady non-Beltrami field, the *vorticity* fields of the solutions have plenty of invariant tori. Indeed, those vorticity fields of the solutions approach the integrable vorticity field of the steady flow. (The vortex form of the Euler equation is more suitable for this consideration, since the vorticity, unlike the velocity, is frozen into the flow.) Similarly, for the Navier–Stokes equation the steady flows close to the Beltrami ones have many invariant tori.

5.C. *Spectrum of the shortened equation*

For a more detailed analysis of solutions of equation (5.2) (and another viewpoint at Proposition 5.2), we expand s into a Fourier series in terms of φ, using the following notation. Let m, which we shall call the *wave vector*, be a pair of integers m_1 and m_2. We denote $m_1\varphi_1 + m_2\varphi_2$ by (m, φ), the number $\sqrt{m_1^2 + m_2^2}$ by $|m|$, and the pair $n_1 = -m_2$ and $n_2 = m_1$ by n.

For each wave vector we determine the "longitudinal," "transverse," and "normal" vector fields

$$e_m = \frac{m_1}{|m|}\frac{\partial}{\partial\varphi_1} + \frac{m_2}{|m|}\frac{\partial}{\partial\varphi_2}, \quad e_n = -\frac{m_2}{|m|}\frac{\partial}{\partial\varphi_1} + \frac{m_1}{|m|}\frac{\partial}{\partial\varphi_2}, \quad e_z = \frac{\partial}{\partial z}.$$

(For $m = 0$ we assume, e.g., $e_m = \partial/\partial\varphi_1$ and $e_n = \partial/\partial\varphi_2$.)

The Fourier expansion of a field s can now be written as

$$s = \sum_m (A_m e_m + B_m e_n + C_m e_z)e^{i(m,\varphi)},$$

where A_m, B_m, and C_m are functions of z.

It can be readily verified that the vector fields e_m, e_n, and e_z have zero divergence with respect to the volume element $d\varphi_1 d\varphi_2 dz$. Hence,

$$\text{div } s = \sum_m (i|m|A_m + \partial_z C_m)e^{i(m,\varphi)} \quad \left(\partial_z := \frac{d}{dz}\right).$$

Consequently, the divergence-free fields are determined by the condition $i|m|A_m + \partial_z C_m = 0$ satisfied for all m.

By virtue of this condition, the set of functions B_m and C_m (for $m = 0$, we have $C_0 = \text{const}$, but A_0 is to be added) can be taken as the "coordinates" in the space

of all fields. In this coordinate system equation (5.2) decouples into a series of triangular systems

$$(5.5) \qquad \begin{cases} \dot{B}_m = -i|m|v_m B_m + v'_n C_m, \\ \dot{C}_m = -i|m|v_m C_m, \end{cases}$$

where $v = v_m e_m + v_n e_n$ is the velocity field of the steady flow (for $m = 0$ we add the equation $\dot{A}_0 = v'_0 C_0$); the prime and the dot denote differentiation with respect to z and t, respectively.

Formula (5.5) again implies the nonexponential instability of equation (5.2) (and proves Proposition 5.2). Furthermore, it determines the spectrum of the latter equation: To each wave vector m one associates a segment of the continuous spectrum along the imaginary axis. The related "frequencies" $|m|v_m$ are equal to all kinds of frequencies (m, v) of the stationary flow on the tori, corresponding to various values of the z-coordinate. The multiplicity of each segment is not less than two (the B- and C-components have the same frequencies).

5.D. The Squire theorem for shear flows

Though the coordinates introduced above are suitable for analyzing the shortened equation (5.2), analysis of the complete equation (5.1) is generally difficult, since in curvilinear coordinates the operator curl^{-1} is of a complicated form. A particular case in which the analysis can be reduced to a one-dimensional problem is that of a flow with straight streamlines. All plane rectilinear flows, as well as the more general ones in which the fluid particles move in parallel planes at constant velocity, which varies in magnitude and direction when passing from one plane to another, belong to this class. Study of the latter may be considered as an approximate analysis of a generic flow in the torus geometry, in which the torus curvature is neglected, while the shear (i.e., the variation of the direction of the streamlines from one torus to another) is taken into consideration.

Let φ_1, φ_2, and z be Cartesian coordinates and the length element $d\ell^2 = d\varphi_1^2 + d\varphi_2^2 + dz^2$. Let $v = v_m e_m + v_n e_n$ be the velocity field of a shear (rectilinear) flow in three-dimensional space (or in a three-torus, whose curvature is neglected).

Proposition 5.4. *The rectilinear three-dimensional flow is exponentially unstable if and only if at least one of the two-dimensional flows of a perfect fluid obtained by the substitution for the velocity vector v of its longitudinal component v_m is exponentially unstable.*

Thus, the problem of exponential instability of the considered class of three-dimensional flows of a perfect fluid is reduced to a similar problem for a set of the two-dimensional flows corresponding to different values of the wave vector m. In the particular case of a nonshear flow (i.e., with a constant direction of the velocity v), all velocity profiles are proportional to each other, and the obtained result agrees with the Squire theorem for a perfect fluid [Squ].

Proof. In this case it is expedient to consider periodic flows of not necessarily 2π-periodicity (e.g., we can assume the periods of φ_1 and φ_2 to be $2\pi X_1$ and $2\pi X_2$, respectively). The only alteration to be introduced in the formulas of Section 5.C is that now the wave vector m runs not through the lattice of integral points but through the lattice $\{(m_1/X_1, m_2/X_2)\}$.

Under these assumptions, the expansion of the vortex field w in terms of the unit vectors e_m, e_n, and e_z is of the form $w = -v'_n e_m + v'_m e_n$. The matrices of the operator curl in the coordinates B_m, C_m, and of the operator corresponding to the Poisson bracket containing w are, respectively,

$$i|m| \begin{pmatrix} 0 & -\text{Id} + |m|^{-2}\partial_z^2 \\ \text{Id} & 0 \end{pmatrix} \quad \text{and} \quad -\begin{pmatrix} i|m|v'_n & v''_m \\ 0 & i|m|v'_n \end{pmatrix},$$

where Id is the identity transformation. Hence, in our coordinates the linearized Euler equation (5.1) decomposes into the systems of equations corresponding to various m. After some calculation, we obtain for $m \neq 0$ the triangular system

$$(5.6) \quad \begin{cases} \dot{B}_m &= \left(i|m|v_m + \frac{v''_m}{i|m|}(id - |m|^{-2}\partial_z^2)^{-1}\right) B_m, \\ \dot{C}_m &= i|m|v_m C_m + v'_n(id - |m|^{-2}\partial_z^2)^{-1} B_m, \end{cases}$$

and for $m = 0$, we have the system $A_0 = B_0 = C_0 = 0$. The first equation contains the B-component only. If the B-component does not have exponential instability, neither does the C-component (this is implied by the nonhomogeneous linear equation obtained for C_m). Finally, note that the equation for B_m contains only the longitudinal velocity component v_m. $\square$

The Jordan form of system (5.6) indicates that in three-dimensional incompressible flows, unlike the two-dimensional ones, the linear increase of vortex perturbations with time is typical, even in the absence of exponential instability. Notice also that equation (5.6) is the same as that derived in the analysis of the two-dimensional flow of a perfect fluid whose velocity profile is the component $v_m(z)$ of the velocity vector of a three-dimensional flow in the direction of the wave vector m.

5.E. Steady flows with exponential stretching of particles

In this section we will define a steady flow of an incompressible fluid for which the velocity field is Beltrami; i.e., it is proportional to its own vorticity, and the field does not have a family of invariant surfaces, as mentioned in Section 5.B. This simple example plays a key role in many other constructions of ideal hydrodynamics and of dynamo theory discussed in the sequel (see, e.g., Section V.4).

Imagine an ideal fluid filling a three-dimensional compact manifold M constructed in the following way. First consider the Euclidean three-dimensional space with coordinates x, y, z and define the following three diffeomorphisms

of the space:

$$T_1(x, y, z) = (x + 1, y, z), \qquad T_2(x, y, z) = (x, y + 1, z),$$
$$T_3(x, y, z) = (2x + y, x + y, z + 1).$$

Each of these transformations maps the integer lattice in the space x, y, z into itself. Identify all points of xyz-space that can be obtained from each other by the successive application of T_i and T_i^{-1} (in any order). The resulting compact analytic manifold M may be thought of as the product of a two-dimensional torus $\{(x, y) \bmod 1\}$ by the segment $0 \le z \le 1$, whose end-tori are identified by means of the formula $(x, y, 0) \equiv (2x + y, x + y, 1)$.

To equip the manifold M with a Riemannian metric, we define a metric in xyz-space invariant with respect to all T_i. We first examine the linear transformation of the xy-plane given by the matrix A ("cat map," Fig. 19):

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \text{i.e.,} \qquad A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + y \end{pmatrix}.$$

The operator A has the eigenvalues $\chi_{1,2} = (3 \pm \sqrt{5})/2$. Note that $\chi_1 > 1 > \chi_2 > 0$, $\chi_1 \cdot \chi_2 = 1$, and the eigendirections are orthogonal to each other. Let (p, q) be a Cartesian system of coordinates in the xy-plane with the axes p and q directed along the eigenvectors with the eigenvalues $\chi_1 > 1$ and $\chi_2 < 1$, respectively.
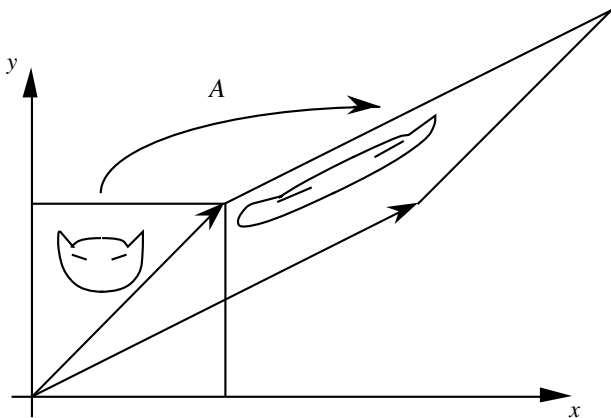
FIGURE 19.   The cat map A of the torus onto itself.

Set the metric to be

$$(5.7) \qquad d\ell^2 = e^{-2\beta z} dp^2 + e^{2\beta z} dq^2 + dz^2, \qquad \text{where} \quad \beta = \ln \chi_1.$$

The metric $d\ell^2$ is invariant with respect to the transformations T_i, and therefore it defines an analytic Riemannian structure on the three-dimensional compact manifold M.

Now consider the vector field grad $z = \partial/\partial z$ in xyz-space. Since it is invariant with respect to the transformations T_i, it descends to a vector field v on the Riemannian manifold M. The field v is harmonic on M: div $v = 0$, curl $v = 0$. Hence, v can be taken as the velocity field of a stationary potential flow of an ideal fluid. Every particle of the fluid moving along that field is stretched exponentially in the q-direction, and it is squeezed in the p-direction, as implied by formula (5.7).

5.F. Analysis of the linearized Euler equation

The Euler equation (5.1), linearized at v, is equivalent to the shortened equation (5.2), since the flow under consideration is potential. The simple geometry of the flow v allows one to solve the latter equation by using formula (5.3). It is convenient to express the solution in the following form. Consider the vector fields

$$e_p = e^{\beta z}\frac{\partial}{\partial p}, \quad e_q = e^{-\beta z}\frac{\partial}{\partial q}, \quad e_z = \frac{\partial}{\partial z}$$

in pqz-space. These fields are invariant with respect to all transformations T_i, and hence, they can be regarded as vector fields on the manifold M. The directions of the fields e_p, e_q, and e_z are invariant with respect to the phase flow g^t of the field e_z (in coordinate form $g^t(p, q, z) := (p, q, z + t)$), while the fields themselves are transformed as follows:

$$g^t_* e_p = e^{-\beta t} e_p, \quad g^t_* e_q = e^{\beta t} e_q, \quad g^t_* e_z = e_z$$

(this explains the names of the *stretching direction* e_q, the *compressing direction* e_p, and the *neutral direction* e_z). Every vector field u on M can be decomposed in these directions,

$$u = u_p e_p + u_q e_q + u_z e_z,$$

where u_p, u_q, and u_z are functions on the manifold M.

Formula (5.3) applied to the stationary flow $v = e_z$ has the form

(5.8) $s_p(t) = e^{-\beta t} U^t s_p(0), \quad s_q(t) = e^{\beta t} U^t s_q(0), \quad s_z(t) = U^t s_z(0),$

where U^t is a linear operator acting on functions on the manifold M by the formula $(U^t f)(a) = f(g^{-t}a)$ for any point $a \in M$. Note that the operator U^t is unitary, since the flow g^t preserves the volume element.

Formulas (5.8) provide rather complete answers to all kinds of questions on the growth of perturbations of the steady flow v. First, they show that the q-component of any vortex perturbation exponentially increases with time, while the p-component decays exponentially.

Further, the spectrum of the operator U^t can be easily analyzed by the Fourier series expansion in terms of (x, y) with fixed z, and for functions independent of x and y by such expansion in terms of z. This spectrum has a countably multiple continuous (Lebesgue) component along the unit circle in $\mathbb{C}$, and also a discrete set of eigenvalues corresponding to the eigenfunctions $\varphi_k(z) = e^{2\pi i k z}$ (k are integers). This implies that the Euler equation (5.1) linearized at the stationary flow $v = e_z$

has a countable set of the (unstable) eigenvalues $\alpha - 2\pi i k$, related to the countable set of increasing perturbations of the vorticity $s = \varphi_k(z)e_q$ $(k = \pm 1, \pm 2, \dots)$.

The difficulty of predicting solutions of the linearized Euler equation (5.1) for flows with the exponential stretching of particles is also indicated by formulas (5.8): To find an approximate solution, it is necessary to know, with considerable precision, a number of high-order harmonics of the initial perturbation $s(0)$, which rapidly increase with t. Formulas (5.8) and (5.4) show that the exponential particle stretching increases drastically the difficulty of predicting the perturbation growth, as compared to the flows defined by the "generic" stationary solutions of the Euler equation with the linear stretching of particles (see Sections 5.B–5.D).

Phenomena similar to those outlined in this example are also encountered in other flows with exponentially stretched particles, e.g., in the *ABC* flows

$$v_x = A \sin z + C \cos y, \ v_y = B \sin x + A \cos z, \ v_z = C \sin y + B \cos x$$

(see Sections II.1.A, V.4.B, and [Hen, Dom] for a study of symmetries and results of computer simulations) or in the geodesic flows on surfaces of negative curvature (see Section V.4.D).

5.G. Inconclusiveness of the stability test for space steady flows

In Section 4.A we gave a sufficient condition for stability of planar fluid flows. Unlike the two-dimensional case, the second variation of the kinetic energy of a stationary flow among isovorticed fields is never sign definite in higher dimensions. It implies that the sufficient stability criterion, based on the second variation, is inconclusive (see Remark 3.7): Quadratic Hamiltonians of a saddle type can govern both stable and unstable flows. This study is based on the consideration of rapidly oscillating perturbations of the steady flow.

Theorem 5.5. *Let M be a three-dimensional closed manifold and v be a steady Euler flow. If* curl *v is not identically zero, then the spectrum of the quadratic form $\delta^2 E$ (i.e., of the corresponding self-adjoint operator) on the tangent space to the coadjoint orbit of v is neither bounded from below nor from above.*

Remark 5.6. This theorem, along with its higher-dimensional version formulated below, has been proved in [S-V]. Indefiniteness of the second variation $d^2 E$ for the 3-D case was earlier established in [Rou1] (and hinted at already in [Arn4]; see also [A-H], where the consideration was put forward for a generic equilibrium in the 3-D case). The main idea underlying all the proofs is that the form $\delta^2 E$ is a sum of two terms, one of which is always positive, but of smaller order than the other. Picking the rapidly oscillating variation ξ, one can explicitly compute the asymptotic expression for $\delta^2 E$ and thus obtain an arbitrary sign for the second variation in the direction ξ.

The unboundedness of the spectrum of the second variation holds for the higher-dimensional generalization of the Euler equation as defined in Section I.7. Namely, let M be an n-dimensional smooth Riemannian manifold ($n \geq 3$) endowed with a volume form μ, and $v^\flat$ the one-form on M obtained from a μ-divergence-free vector field v by means of the identification $v^\flat(w) = (v, w)$ determined by the Riemannian metric (,). The kinetic energy is given by $E(v) = \frac{1}{2}\langle v, v \rangle = \frac{1}{2}\int_M (v, v)\mu$.

Theorem 5.5′ [S-V]. *Let u be a smooth steady solution of the Euler equation in M. The second variation $\delta^2 E$ of the energy among the isovorticed vector fields is identically zero, whereas $v^\flat$ is locally "potential" in the sense that $d(v^\flat) \equiv 0$. Otherwise, the spectrum of the self-adjoint operator corresponding to the quadratic form $\delta^2 E$ on the space of isovorticed fields is neither bounded from below nor from above.*

Remark 5.7. Actually, the Euler equation is defined on cosets of 1-forms on M: $[v^\flat] \in \Omega^1(M)/d\Omega^0(M)$ (see Chapter I). There are as many cosets furnishing the condition $d[v^\flat] = 0$ as elements in $H^1(M)$, i.e., a finite-dimensional space. Hence, among all stationary flows on the manifold M, there are exactly $b_1(M) := \dim H^1(M)$ linearly independent ones for which the second variation of the kinetic energy is zero. For all other steady flows this variation is indefinite.

Lemma 5.8. *The second variation of the energy $E(v) = \frac{1}{2}\langle v, v \rangle = \frac{1}{2}\langle v^\flat, v^\flat \rangle$ on the (image in the Lie algebra of the coadjoint) orbit of the "isovorticed fields" is given by the quadratic form*

$$(5.9) \qquad \delta^2 E(\xi) = \frac{1}{2}\langle i_\xi dv^\flat + dp, i_\xi dv^\flat + dp \rangle + \frac{1}{2}\langle i_\xi dv^\flat + dp, L_\xi(v^\flat) \rangle,$$

where the function p is chosen to make the 1-form $i_\xi dv^\flat + dp$ correspond to a divergence-free field after the Riemannian identification.

Proof of Lemma. The proof is a straightforward application of formula (3.2) to the coadjoint operator $B(v, \xi) = i_\xi d(v^\flat) + dp$. All fields are supposed to be square-integrable. The formal tangent space to the coadjoint orbit of the 1-form $v^\flat$ is the image of the operator B. $\qquad\square$

For a three-dimensional manifold M, this formula reads as

$$\delta^2 E(\xi) = \frac{1}{2}\langle (\nabla \times v) \times \xi + \nabla p, (\nabla \times v) \times \xi + \nabla p \rangle$$

$$+ \frac{1}{2}\langle (\nabla \times v) \times \xi + \nabla p, \nabla \times (\xi \times v) \rangle.$$

The operator $B(v, \xi)$ in this case becomes $B(v, \xi) = (\nabla \times v) \times \xi + \nabla p$, where the pressure function p is chosen to make the vector field $(\nabla \times v) \times \xi + \nabla p$ divergence free.

Proof of Theorem 5.5′. Certainly, $dv^\flat \equiv 0$ implies $dp = 0$, and hence, $\delta^2 E(\xi) \equiv 0$.

Assume now that the 2-form $dv^\flat$ and the vector field v are both nonzero at a point $x_0 \in M$. Fix some function $\varphi(x)$ for which $(v, \nabla\varphi)$ and $d\varphi \wedge dv^\flat$ are both nonzero in a neighborhood $\mathcal{U}$ of x_0. Pick smooth vector fields a_R and a_I that are orthogonal to $\nabla\varphi$ everywhere, vanish outside $\mathcal{U}$, and obey the inequalities $du^\flat(a_R, a_I) \geq 0$ everywhere, and $du^\flat(a_R, a_I) > 0$ in a smaller neighborhood $\mathcal{U}' \subset \mathcal{U}$. Finally, define a complex vector field $a = a_R + \sqrt{-1}a_I$ (where we use the notation $\sqrt{-1}$ for the imaginary unit to distinguish it from the operator i_v).

Our goal is to construct deformations ξ_ε (uniformly bounded in ε) for which $\delta^2 E(\xi_\varepsilon)$ is arbitrarily large positive or negative. Note that it is enough to choose ξ_ε to be a complex vector, if we extend the operator $\delta^2 E$, as well as the Hermitian inner product, to the complexification of the space of vector fields on the manifold M. Indeed, consider the Hermitian inner product $\langle\,,\,\rangle_\mathbb{C}$, linear in the first argument and antilinear in the second, that extends the real inner product $\langle\,,\,\rangle$ on the vector fields. Then boundedness of the spectrum of $\delta^2 E$ implies that the *real part* of the value $\langle(\delta^2 E)\xi_\varepsilon, \xi_\varepsilon\rangle_\mathbb{C}$ is bounded both from below and from above whenever ξ_ε belongs to some fixed ball in the Hilbert space of square-integrable complex vector fields.

To construct such deformations ξ_ε, consider for simplicity the case where μ is the Riemannian volume form on the manifold. Then a one-form $u^\flat$ corresponds to a divergence-free vector field u if and only if $d^*(u^\flat) \equiv 0$ (where the operator $d^* : \Omega^k(M, \mathbb{C}) \to \Omega^{k-1}(M, \mathbb{C})$ is dual to the exterior derivative operator $d : \Omega^k(M, \mathbb{C}) \to \Omega^{k+1}(M, \mathbb{C})$ by means of the identification of $\Omega^k(M^n, \mathbb{C})$ and $\Omega^{n-k}(M^n, \mathbb{C})$ provided by the metric).

Define the rapidly oscillating vector fields ξ_ε as the following $O(\varepsilon)$-correction of the field $a \cdot \exp(\sqrt{-1}\varphi/\varepsilon)$ to make it divergence free: ξ_ε is dual to the 1-form

$$\xi_\varepsilon^\flat = \varepsilon\sqrt{-1}\,d^*\left(\frac{d\varphi \wedge a^\flat}{\|d\varphi\|^2}\exp(\sqrt{-1}\varphi/\varepsilon)\right) = a^\flat \exp(\sqrt{-1}\varphi/\varepsilon) + O(\varepsilon).$$

Then the leading term of $\delta^2 E(\xi_\varepsilon)$ in the ε-expansion as $\varepsilon \to 0$ is

$$\delta^2 E(\xi_\varepsilon) = \frac{1}{2\varepsilon}\langle i_{\xi_\varepsilon}du^\flat + dp, \sqrt{-1}(u, \nabla\varphi)\,a^\flat \exp(\sqrt{-1}\varphi/\varepsilon)\rangle_\mathbb{C} + O(1)$$

$$= -\frac{\sqrt{-1}}{2\varepsilon}\int_M (u, \nabla\varphi)\,du^\flat(a, \bar{a})\mu + O(1)$$

$$= -\frac{1}{\varepsilon}\int_M (u, \nabla\varphi)\,du^\flat(a_R, a_I)\mu + O(1),$$

where $\langle\,,\,\rangle_\mathbb{C}$ is the Hermitian inner product, extending the real inner product $\langle\,,\,\rangle$.

By assumption, the inner product $(u, \nabla\varphi)$ is nonzero on $\mathcal{U}$, while $du^\flat(a_R, a_I)$ is positive in $\mathcal{U}'$ and nonnegative otherwise. Hence the integral is nonzero. Therefore, we can make the real part of $\delta^2 E(\xi_\varepsilon)$ arbitrarily large positive or negative by choosing ε to be of appropriate sign and sufficiently close to zero. Thus, $\delta^2 E$ is not a sign-definite form, and it has a spectrum unbounded in both directions. □

Remark 5.9 [S-V]**.** For a manifold with boundary the same conclusion holds. One can take ξ_ε vanishing near the boundary and obtain arbitrarily large negative or positive values of $\delta^2 E(\xi_\varepsilon)$. The domain of the corresponding self-adjoint operator $\delta^2 E$ contains all smooth divergence-free vector fields with compact support in the interior of M.

Remark 5.10. One can argue that indefiniteness of the second variation is indicative of instability (see, e.g., [A-H]). Though the sufficient criterion discussed above says nothing in this case, other methods can be applied to certain flows (see [Yu5] for an interesting discussion, [Vla2] for the direct Lyapunov method and [FGV, FV1] for an instability criterion valid for some particular three-dimensional flows).

For instance, a fluid possessing surface tension and filling an upside-down cylindrical glass (with any cross section) is shown to be unstable [VlB, Vla2]. To the best of our knowledge, there is no proof of (actual nonlinear) instability if the shape of the container is not cylindrical.

The situation changes slightly for the system of MHD equations. In contrast with the purely hydrodynamical setting, it is possible to obtain three-dimensional examples of MHD equilibria for which the second variation of the total energy is definite [FV2]. The class of flows whose stability may be determined by the sufficient criterion discussed in this and preceding sections is very restricted. In particular, the second variation of energy turns out to be indefinite for the flows having a point where the vectors of the velocity v and of the vorticity curl v are nonzero and nonparallel to the vector of the magnetic field B. The same statement holds for fields with parallel magnetic and velocity fields if the magnetic field is weak enough: $\|v\| > \|B\|$ at some point [FV2]. Other applications of the stability analysis to MHD can be found in [VlM, VMI]. Stability of steady two- and three-dimensional flows of an ideal fluid with a free boundary was studied in [SYu]; for the stability analysis of stratified ideal, barotropic, and other fluids see [Dik, A-H, HMRW, Gri, Vla3]. We also refer to [Arn14, DoS, FV1, Lif, Shf] for various stability and asymptotic results for perturbations of steady solutions of the Euler and Navier–Stokes equations.

§6. Features of higher-dimensional steady flows

The existence of the Bernoulli function for a steady fluid flow is a general phenomenon valid for any dimension (see Section 1.A). In this section we discuss (following [GK1, 2]) the consequences of the presence of this extra first integral for steady solutions of the Euler equation of an ideal fluid in higher dimensions.

6.A. Generalized Beltrami flows

Let v be an analytic divergence-free field of a steady flow on an odd-dimensional compact manifold M^{2n+1} equipped with a volume form μ.

Definition 6.1. A trajectory of the field v is called *chaotic* if it is not contained in any analytic hypersurface in M^{2n+1}.

For instance, a generic trajectory of an ergodic flow is chaotic.

Proposition 6.2 (=1.8', [GK2]). *An analytic steady field v with at least one chaotic trajectory is proportional to its vorticity ξ; i.e., $\xi = C \cdot v$, where $C \in \mathbb{R}$.*

Remark 6.3. Recall that in the odd-dimensional case the vorticity field is defined by the relation $i_\xi \mu = \omega^n$, where the two-form $\omega = du$ is the differential of the one-form u dual to the vector field v: $u(\cdot) = (v, \cdot)$; see Chapter I. Thus, by the proposition, the field v with a chaotic trajectory is an "eigenvector" of the operator curl: $v \mapsto \xi$, even though for $n > 1$ this operator is nonlinear! It is natural to call such a field v a *generalized Beltrami* flow. The theorem manifests that higher-dimensional Beltrami flows, as well as the three-dimensional ones, have quite a complicated structure. In particular, the mixing in a steady flow might occur only if at least one chaotic trajectory exists, i.e., only for the generalized Beltrami flows. On the contrary, a non-Beltrami steady flow is fibered by a family of hypersurfaces invariant under the flow, and therefore actual mixing for such a flow is impossible. The proof of the theorem closely follows the argument used for the three-dimensional case in [Arn3, 4]; cf. Section 1.A.

Proof. The vorticity field ξ commutes with the velocity field v for any steady flow (see Remark 1.4). The fields ξ and v are both tangent to the "Bernoulli surfaces," i.e., to the level hypersurfaces of the analytic Bernoulli function $\alpha = p + i_v u$, which is defined by the stationary Euler equation $i_v du = -d\alpha$.

If the Bernoulli function α is nonconstant, then trajectories of v lie on level hypersurfaces of α, which contradicts the assumption. (Note that similar to the three-dimensional case, the nonsingular Bernoulli surfaces ($d\alpha \neq 0$) have zero Euler characteristic, since the tangent field v has no singular points on them.) If the function α is constant, then the fields ξ and v are collinear (Remark 1.6). Consider the function $\kappa := v^2/\xi^2$ (or, alternatively, $(\xi, \xi) = \kappa \cdot (v, v)$). Owing to the commutativity of ξ and v, the function κ is invariant under the flow of v. Therefore, the field v is tangent to the level surfaces of κ. Since v has a chaotic trajectory, the only possibility remaining is that $\kappa \equiv$ const. (Note that the Bernoulli function α is analytic, and the function κ is the ratio of two analytic functions.) Hence the functions α and κ are both constant, and the fields ξ and v are locally proportional: $\xi = C \cdot v$, where $C = \pm 1/\sqrt{\kappa} =$ const. $\qquad\square$

Example 6.4. The Hopf vector field $(x_2, -x_1, x_4, -x_3, \ldots, x_{2n+2}, -x_{2n+1})$ is an example of an eigenvector field for the curl operator on $S^{2n+1} \subset \mathbb{R}^{2n+2}$ without chaotic trajectories. The theorem above claims that the existence of such a trajectory makes the vector field an "eigenvector" of curl. It would be interesting to find a nontrivial example of a higher-dimensional ABC flow and to compare its ergodic properties with those in the three-dimensional case (see, e.g., [Hen]).

In particular, one wonders if there is an analogue for higher dimensions of the analytic nonintegrability of certain ABC flows, proved in [Zig2].

6.B. Structure of four-dimensional steady flows

The main result of this section shows that the steady flows of a four-dimensional fluid are very similar to integrable Hamiltonian systems with two degrees of freedom.

Here and below we deal with an even-dimensional orientable Riemannian manifold M^{2n} endowed with a volume form μ. In this case, a generic steady solution v gives rise to the closed 2-form $\omega = du$, which is symplectic (nondegenerate) almost everywhere on M. In particular, it allows one to define another (besides α) invariant function on the manifold: $\lambda(x) = \omega^n/\mu$, called the vorticity function (or "symplectic volume" element). The function λ is invariant, since $L_v\omega = 0$ and $L_v\mu = 0$. This means that the vorticity function λ and the Bernoulli function α are first integrals of the flow of v on M.

Let $\rho = (\alpha, \lambda) : M \to \mathbb{R}^2$ and Γ be the set formed by all $x \in M$ such that either $\lambda(x) = 0$ or $\rho(x)$ is a critical value of ρ. In other words, Γ is the union of the zero λ-level Λ and of the preimage of the set of critical values of ρ.

Theorem 6.5 [GK1, 2]. *Let M be a closed orientable four-dimensional manifold. Then*

(1) *the open set $U = M \setminus \Gamma$ is invariant under the flow of v;*
(2) *every connected component of U is fibered into two-dimensional tori invariant under the flow; and*
(3) *on each of these tori the flow lines are either all closed or all dense.*

Proof. The form ω is symplectic on the complement to the set $\Lambda = \{\lambda = 0\}$. The vector field v is Hamiltonian (relative to this symplectic form) with the Hamiltonian function α: by definition $i_v\omega = -d\alpha$. Let ξ be the Hamiltonian vector field on $M \setminus \Lambda$ with the Hamiltonian λ. Observe that the Poisson bracket of the functions α and λ is identically zero on $M \setminus \Lambda$, since $\{\alpha, \lambda\} = L_v\lambda = 0$. Therefore, the fields v and ξ commute, and their flows together give rise to an $\mathbb{R}^2$-action on $M \setminus \Lambda$. The map ρ is, in fact, the momentum mapping for this action. The map ρ is invariant with respect to the action, and the orbits coincide with the connected components of ρ-levels. The projection $\rho|_U : U \to \rho(U)$ is a proper submersion, since defining U we have excluded from M all critical points of ρ. Hence each orbit in U is a smooth closed surface, and so it is either a torus or a Klein bottle. Furthermore, this surface is cooriented by $d\alpha \wedge d\lambda$. As a result, we see that the surface is orientable, i.e., a torus. Therefore, ρ fibers every connected component of U into tori.

On each orbit, the flow of ξ acts transitively on integral curves of v. Moreover, the field ξ does not have zeros on U since its Hamiltonian function λ does not

have critical points there. Thus the integral curves of v, on which ξ acts, are either all closed or all dense on each torus.   □

Note that for a "generic" pair of α and λ the set U is open and dense in M. Thus the theorem gives an almost complete description of the flow of v.

The real-analytic version of the latter theorem for a manifold without boundary now looks as follows.

Theorem 6.6 [GK2]. *Let M be as in the theorem above. Assume, in addition, that all the data (i.e., M, μ, and the metric), as well as ω, are real-analytic, and $d\alpha \wedge d\lambda \neq 0$ somewhere on M. Then Γ is a semianalytic subset nowhere dense in M, and $U = M \setminus \Gamma$ has a finite number of connected components. Every connected component is fibered into two-dimensional tori invariant under the flow. On each of these tori the flow lines are either all closed or all dense.*

A version of this theorem holds for a manifold M with boundary (see [GK2] for more detail).

Remarks 6.7. (A) For an arbitrary even-dimensional manifold M^{2n}, we can assert that M is a union of $(2n - 2)$- (or less) dimensional submanifolds, such that the steady vector field v is tangent to them. These submanifolds are obtained as intersections of the levels $\alpha = $ const and $\lambda = $ const and have zero Euler characteristic.

(B) For an arbitrary odd-dimensional M^{2n+1}, instead of the function $\lambda = \omega^n/\mu$ (and of the covector field $d\lambda$), we define the vorticity vector field ξ by $i_\xi \mu = \omega^n$. The fields ξ and v commute and thus give rise to an $\mathbb{R}^2$-action on M^{2n+1}. So in this case a steady flow gives rise to a foliation of *dimension* 2, unlike the foliation of *codimension* 2 in the even-dimensional case.

6.C. Topology of the vorticity function

Let ω be the two-form associated to a stationary divergence-free solution v on M^{2n} (i.e., $\omega = du$, where u is the differential 1-form $u(\cdot) = (v, \cdot)$ defined by the Riemannian metric $(\ ,\ )$ on M). In this section, we study the topology of the vorticity function $\lambda = \omega^n/\mu$ of the steady flow v. We describe some special features of such λ that the pair (λ, ω) (under a mild condition) does not admit "too many symmetries."

Let $\mathfrak{g}$ be the Lie algebra of all divergence-free vector fields on M. Steady flows are critical points of the energy on the coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ that consists of the 2-forms associated to the fields on M isovorticed with v. It is clear that topological invariants of λ, such as the number of its critical points and their indices, depend only on the orbit $\mathcal{O}$. This simple observation will enable us to find orbits with no stationary solutions at all (see Section 6.D).

Definition 6.8. A function f on a compact symplectic manifold (P, ω) *does not admit extra symmetries* if an arbitrary function g satisfying $\{f, g\} = 0$ is constant on connected components of the level sets of f (i.e., $\{f, g\} = 0$ implies that the differential dg is proportional to df with coefficient depending on the point on P).

Remark 6.9. On a two-dimensional symplectic manifold no functions admit extra symmetries. Conjecturally, a generic function on a compact symplectic manifold of any dimension does not admit extra symmetries. It is true for dim $M = 4$ (cf. [MMe]). The question turns out to be closely related to some subtle problems in Hamiltonian dynamics. The general conjecture can be regarded as a Hamiltonian version of the following problem of generic nonintegrability.

Remark 6.10: Digression on nonintegrability. From the time of Poincaré one usually has used the term "a nonintegrable dynamical system" in the sense of "a dynamical system having no analytic first integrals." However, there exists a number of other possibilities. For instance,

(1) the absence of invariant hypersurfaces (or of principal ideals),
(2) the absence of invariant closed 1-forms (or of multivalued first integrals),
(3) the absence of invariant distributions of tangent subspaces (or of invariant Pfaff modules), and
(4) the absence of invariant foliations (or of invariant completely integrable Pfaff systems).

Consider a dynamical system with discrete time (a diffeomorphism of a compact manifold) and an object of one of the above types (a function, an ideal, a closed 1-form, etc.) The images of this object under the iterations of the diffeomorphism may form a finite set (if they are repeated periodically) or an infinite sequence and may generate a finite-dimensional or infinite-dimensional space. These properties reflect the "degree of chaoticity" of the dynamical system.

Problem 6.11. Do the nonintegrable systems (in the sense of each of the four definitions above) form an open set in the space of dynamical systems on manifolds of sufficiently high dimension? Conjecturally, this is the case in the space of Hamiltonian systems near an elliptic equilibrium point.

Even specific examples of systems that are nonintegrable in the strong sense ((1),(2),(3), or (4)) would be interesting. The following example of chaotic behavior is due to Kozlovsky [Koz1]. Consider a germ of an analytic mapping

$$z \mapsto e^{i\theta} z + z^2$$

of the complex line $z \in \mathbb{C}$ to itself in a neighborhood of the (elliptic) fixed point 0. Let an irrational θ be unusually well approximated by rational numbers. Then

there are infinitely many periodic trajectories in any neighborhood of the origin. Such mappings are nonintegrable in the sense of (1)–(4).

One more extension of the integrability property has been suggested by Yudovich [Yu2]. He introduced the notion of cosymmetry of a vector field. A *cosymmetry* is a field of hyperplanes in the tangent spaces containing the given vector field (one might call them nonholonomic constraints). This field of hyperplanes is allowed to degenerate at some points of the manifold, and it is defined by a 1-form (possibly with zeros) annihilated at every point by the given vector field.

Every nonzero vector field has locally some trivial cosymmetries. The existence of a global cosymmetry implies some restrictions on the topological properties of the field. Example: If a field with an equilibrium has a nontrivial cosymmetry, then the equilibrium is nonisolated (and generically belongs to a curve of equilibria). If a vector field admits two cosymmetries, it generically has a surface of equilibria, etc. This phenomenon is described by a "cosymmetric version" of the implicit function theorem [Yu2, KuY]. Furthermore, for dynamical systems with cosymmetries one observes generic bifurcations of an equilibrium point into a family of those points (the phenomenon of infinite codimension among all dynamical systems).

Yudovich has discovered nontrivial cosymmetries in some physical problems of hydrodynamical origin (fluid convection in porous media) and of Newtonian mechanics. For instance, if a vector field has a first integral ϕ, then the differential $d\phi$ is a (holonomic) cosymmetry. (Example: For Newton's second law $\ddot{x} = F(x)$ with a *potential* force $F(x)$, the sum of the kinetic and potential energy is the first integral of the equation.) The notion of cosymmetry provides a natural framework for the validity of the result of the Noether theorem on the existence of momentum-like first integrals for the Newton equation $\ddot{x} = F(x)$ with a *nonpotential* force $F(x)$ [Yu2]. The nonholonomic cosymmetries of this equation ensure (generically) the existence of continuous families of equilibria even for this classical situation.

Returning to steady fluid flows in even dimensions, we need the following:

Definition 6.12. A coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ *does not admit extra symmetries* if for any (or, equivalently, for some) 2-form $\omega \in \mathcal{O}$ the corresponding vorticity function λ does not admit extra symmetries on $\lambda^{-1}([a, b])$ for any pair of its regular values $0 < a < b$ or $a < b < 0$. (Note that the form ω is symplectic precisely on the complement to the zero level of $\lambda = \omega^n / \mu$.)

Definitions 6.8 and 6.12 are consistent: A function f on a compact symplectic manifold does not admit extra symmetries if and only if its restriction to the pre-image of any segment with regular endpoints does not admit them.

Definitions 6.13. A function on a compact manifold is a *Morse function* if all its critical points are nondegenerate, i.e., the *Hessian matrix* of the second derivatives of the function is nondegenerate at every critical point. The number of negative eigenvalues of the Hessian matrix is called the *Morse index* of the critical point.

An orbit $\mathcal{O} \subset \mathfrak{g}^*$ has *Morse type* if for any (or, equivalently, for some) $\omega \in \mathcal{O}$ the function λ is a Morse function on M constant on every connected

component of ∂M. The orbit is called *positive* if $\lambda(x)$ is positive for all $x \in M \setminus \partial M$.

Theorem 6.14 [GK2]. *Let* $\dim M = 2n \geq 4$ *and* $\mathcal{O}$ *be a Morse-type orbit without extra symmetries. Assume that* $\mathcal{O}$ *contains a steady solution. Then, for every* $\omega \in \mathcal{O}$ *all the critical points of the vorticity function* λ *have indices either no less than n or no greater than n on every connected component of* $M \setminus \{\lambda = 0\}$.

Example 6.15. If $\mathcal{O}$ is as above and $\lambda > 0$ on $M \setminus \partial M$, then λ cannot have both a local maximum (index $2n$) and a local minimum (index 0) on $M \setminus \partial M$.

Proof of Theorem. For simplicity assume that $\mathcal{O}$ is a positive orbit, i.e., $\lambda > 0$ on M. Only a minor modification is required to prove the general case. Let $\omega \in \mathcal{O}$ be a stationary solution ($L_v \omega = 0$) and α the corresponding Bernoulli function such that $d\alpha = -i_v \omega$.

Since $\lambda = \omega^n / \mu$ does not admit extra symmetries and $\{\alpha, \lambda\} = 0$, the function α must be constant on the connected components of λ-levels.

Lemma 6.16. *The functions* λ *and* α *have the same critical points. In particular, the critical points of* α *are isolated.*

Proof of Lemma. Since λ does not admit extra symmetries, $d\lambda(x) = 0$ implies that $d\alpha(x) = 0$. The rest of the critical set of α may only be the union of some connected components of λ-levels. For a vector field v and the Riemannian dual 1-form $u(\cdot) = (v, \cdot)$ one has $u(v) = (v, v) \geq 0$.

Consider the vector field η on M defined by the formula $i_\eta \omega = u$. The field η is expanding for the 2-form $\omega = du$: $L_\eta \omega = \omega$. Furthermore, the field η is gradient-like for the Bernoulli function α:

$$L_\eta \alpha = i_\eta d\alpha = -i_\eta i_v \omega = i_v u = u(v) \geq 0.$$

Moreover,

$$(6.1) \qquad\qquad L_\eta \alpha = 0 \Leftrightarrow u(v) = 0 \Leftrightarrow u = 0.$$

If the critical set of α contains a connected component Γ of a λ-level, then $L_{\eta_x} \alpha = 0$ for all $x \in \Gamma$, and as a consequence of (6.1), $u|_\Gamma = 0$. Hence, $\omega|_\Gamma = du|_\Gamma = 0$. This is impossible, because Γ is a hypersurface in the symplectic manifold (M, ω) and $2n = \dim M \geq 4$. The lemma is proved. $\qquad\square$

Now observe that all zeros of the vector field η are nondegenerate, as follows from $L_\eta \omega = \omega$. Therefore, the field η has smooth complementary dilating and contracting manifolds in a neighborhood of each of its stagnation points. Moreover, the dimension of the dilating manifold for each point must be at least n. Indeed, the restriction of the symplectic form ω to the contracting manifold of η must be zero by virtue of the expanding property of η, and hence all the contracting manifolds have dimension at most n.

Now we are ready to complete the proof of the theorem. The field η is gradient-like for the function α. Therefore, η is either gradient- or antigradient-like for λ on the whole of M, since the λ- and α-levels coincide in a neighborhood of every critical point and λ is a Morse function. Thus, at all critical points of the vorticity function λ the dimensions of all its dilating or of all its contracting manifolds are simultaneously bounded by n from below. This gives the desired inequality for the Morse indices of the λ-critical points.   □

One may prove that all the critical points of α are nondegenerate except, possibly, for its maxima and minima.

Theorem 6.17 [GK2]. *Let M be diffeomorphic to the two-dimensional disk B^2. If a Morse-type orbit $\mathcal{O} \subset \mathfrak{g}^*$ contains a stationary solution, then for any $\omega \in \mathcal{O}$ the vorticity function λ cannot simultaneously have a local maximum and a local minimum in M, provided that $\lambda > 0$ on $M \setminus \partial M$.*

Note that since $\dim M = 2$, the orbit $\mathcal{O}$ does not automatically admit extra symmetries.

The proof below is a formalization of the following argument, which is evident from a physical viewpoint. Minima and maxima of the vorticity function correspond to rotations of the fluid in opposite directions. On the other hand, the positivity of λ prescribes *a priori* a counterclockwise drift.

Proof. First, recall that α cannot have maxima. Indeed, in a neighborhood of a maximum the gradient-like (for η) field η would shrink the area, which contradicts the equation $L_\eta \omega = \omega$. Let Γ be the critical set of α. Observe that since α is constant on ∂M, the set Γ either contains the boundary ∂M or does not meet it. We claim that $M \setminus \Gamma$ is connected. To prove this, assume the contrary. Then there exists an open set $U \subset M \setminus \Gamma$ such that $\partial U \subset \Gamma$. The set U is invariant under the flow of η, since $d\alpha$ (and thus η) vanishes on Γ. On the other hand, as above, the existence of such a set U contradicts the area expansion.

Observe that the field η is gradient-like for λ in a neighborhood of every local minimum of λ: Indeed, every local minimum of λ is a local minimum of α, and the field η is gradient-like for α. Meanwhile, near a local maximum of λ, the field η must be antigradient-like for λ. Switching from being gradient-like to antigradient-like (and vice versa) may occur only on Γ. But Γ does not divide M. Hence η is either gradient-like or antigradient-like on all of M. The theorem follows.   □

6.D. Nonexistence of smooth steady flows and sharpness of the restrictions

Applying Theorems 6.14 and 6.17, one can easily find a coadjoint orbit that does not contain a steady solution.
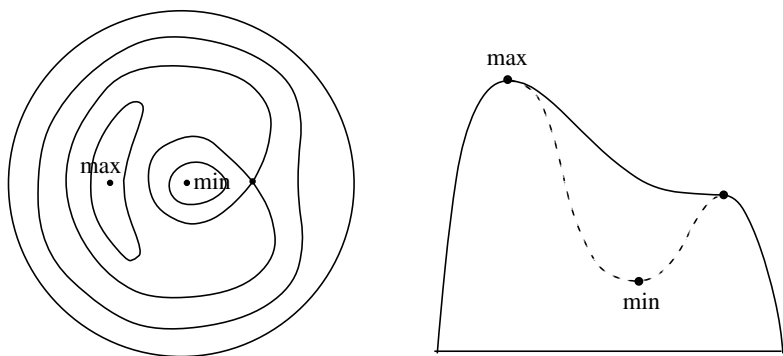
FIGURE 20.   Level curves and a profile of the vorticity function having no smooth steady flow.

The case of a two-dimensional M is particularly simple. Consider a disk $M = B^2 \subset \mathbb{R}^2_{x,y}$ with $\mu = dx \wedge dy$ and $\omega = \lambda \cdot \mu$, where λ is a positive Morse function on B such that $\lambda|_{\partial B} = \text{const}$. Assume also that λ has both a local maximum and a local minimum in the interior of B (see, e.g., Fig. 20).

Corollary 6.18 (of Theorem 6.17). *There is no smooth steady solution on B^2 whose vorticity function is obtained from the function λ by an area-preserving diffeomorphism.*

Note that a "generalized steady solution" with a discontinuous vorticity function may still exist and be of certain interest for applications [Mof4].

Remark 6.19 [GK2]**.**   It turns out that Theorems 6.8 and 6.10 are almost sharp as long as we are not concerned about the metric. Namely, there is no general restriction on the topology of the vorticity function except that given by the theorems.

In the two-dimensional case one can consider, for example, a positive smooth subharmonic function λ on $\mathbb{C} \approx \mathbb{R}^2$, constant on the unit circle. Then on the unit disk B^2 there exists a metric $(\cdot, \cdot)$ and an area form μ such that λ is the vorticity function of a steady solution. In particular, the vorticity function may have saddle critical points, at least for some metrics and volume forms.

A higher-dimensional version of Corollary 6.18 follows from Theorem 6.14. Let $\mathcal{O} \subset \mathfrak{g}^*$ be a Morse-type orbit that is positive (i.e., $\lambda > 0$) and has no extra symmetries.

Corollary 6.20 (of Theorem 6.14). *Assume that for some $\omega \in \mathcal{O}$ the vorticity function λ has a critical point of index $k_1 < n$ and a critical point of index $k_2 > n$, where $2n = \dim M$. Then the coadjoint orbit $\mathcal{O}$ contains no steady solutions.*

$\square$

Corollary 6.21. *Assume that $H^{k_1}(M, \mathbb{R}) \neq 0$ and $H^{k_2}(M, \mathbb{R}) \neq 0$ for some $k_1 < n$ and $k_2 > n$. Then the coadjoint orbit $\mathcal{O}$ contains no steady solutions.*

Proof. The corollary is proved by application of the Morse inequalities. $\square$

Now the sharpness result reads as follows.

Theorem 6.22 [GK2]. *Let M be a compact manifold with boundary, $\dim M = 2n \geq 6$, and λ a smooth positive function on M such that f is constant on connected components of ∂M and all the critical points of λ have indices no greater than n. Assume, in addition, that M admits an almost complex structure. Then there exist a metric and a volume form on M such that λ is the vorticity function of a steady solution.*

The proof uses the result of Ya. Eliashberg [El2] that the manifold M admits a complex structure such that the closed 2-form $\omega = -2 \operatorname{Im} \partial \bar{\partial} \lambda$ is a symplectic form on M.

Various connections between the steady solutions and complex structures, as well as further details and other subtle restrictions on the pairs (ω, λ) imposed by the existence of a steady solution, are discussed in [GK2].

Chapter III

Topological Properties of Magnetic and Vorticity Fields

The interior media of stars and planets are often virtually perfect conductors and possess magnetic fields. These fields are said to be "frozen" into the medium (for instance, plasma or magma) in spite of temperatures of a million degrees. Mathematically this means that any motion of the medium transports the fields' by a diffeomorphism action preserving the mutual position of the fields' trajectories. Such a transform may diminish the field magnetic energy. The topological structure of the field provides obstacles to the full dissipation of the magnetic energy of the star or planet.

On the other hand, inhomogeneity of the medium's motion (e.g., the "differential rotation") stretches the particles and hence might amplify the magnetic energy (transforming part of the kinetic energy of the motion into magnetic energy). This competing mechanism is apparently responsible for the dynamo effect, generating a strong magnetic field from very small magnetic "seeds" (see Chapter V).

§1. Minimal energy and helicity of a frozen-in field

1.A. *Variational problem for magnetic energy*

In this chapter we will look for the energy minimum for the fields obtained from a given divergence-free vector field under the action of volume-preserving diffeomorphisms.

The *energy* of a vector field ξ defined in a domain M of the three-dimensional Euclidean space $\mathbb{R}^3$ is the integral $E = \int_M (\xi, \xi)\mu$. (It differs by a factor of 2 from the energy used in preceding chapters, which simplifies noticeably the estimates below. Throughout Chapter III, the space $\mathbb{R}^3$ is always equipped with the standard metric, and μ is the volume form.)

A more general setting assumes that M is a Riemannian manifold, possibly with boundary. The fields are supposed to be divergence free with respect to the Riemannian volume form (and to obey some boundary conditions, such as

tangency to the boundary of M, or equality of the field normal component at the boundary to a prescribed function). The energy $E = \langle \xi, \xi \rangle = \int_M \langle \xi, \xi \rangle \mu$ is a *geometric* characteristic of the field relying on the choice of the Riemannian metric (,).

Our purpose is to estimate the energy by means of topological features of the field. Here a feature of the field is called *topological* if it persists under the action of diffeomorphisms preserving the volume element (but not necessarily the metric).

Remark 1.1. In magnetohydrodynamics, where this variational problem naturally arises, the role of ξ is played by a magnetic field **B**, frozen into a fluid of infinite conductivity (but of finite viscosity ν) filling a "star" M.

With an appropriate choice of units, the velocity field v and the magnetic field **B** satisfy the system of equations (cf. Section I.10)

$$\begin{cases} \frac{\partial v}{\partial t} + (v, \nabla)v = -\nabla p + \nu \Delta v + (\mathrm{curl}\, \mathbf{B}) \times \mathbf{B}, \ \mathrm{div}\, v = 0, \\ \frac{\partial \mathbf{B}}{\partial t} + \{v, \mathbf{B}\} = 0, \ \mathrm{div}\, \mathbf{B} = 0, \end{cases}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket of two vector fields. The covariant differentiation $(v, \nabla)v$, the Laplace operator $\Delta = -\mathrm{curl}\, \mathrm{curl}$, the vorticity $\mathrm{curl}\, \mathbf{B}$, and the cross product $\times$, standard for $\mathbb{R}^3$, have natural generalizations to the case of an arbitrary Riemannian manifold M. The magnetic field **B** and the velocity field v are prescribed at the initial moment. The term $(\mathrm{curl}\, \mathbf{B}) \times \mathbf{B}$ represents the Lorentz force $\mathbf{j} \times \mathbf{B}$ acting on a current $\mathbf{j}$, which coincides (modulo the factor 4π) with $\mathrm{curl}\, \mathbf{B}$ according to the Maxwell equation.

Physicists suggest that during evolution the kinetic energy dissipates due to the viscosity term $\nu \Delta v$, and the motion ceases "at the end," each particle approaching some terminal position. If this happens, the magnetic field, being frozen-in, will attain some terminal configuration. The energy of this terminal field must be a local minimum; otherwise the magnetic energy would have been converted into kinetic energy, and because of the Lorentz force, the fluid would move further until the hydrodynamical viscosity dissipated the excess of the magnetic energy above the minimum.

1.B. Extremal fields and their topology

The variational principle for magnetic fields is dual to that for the steady fluid flows (studied in Chapter II) in the following sense.

The energy functional that undergoes a minimization procedure is the same in both problems. The domain of this functional in the magnetic case consists of all fields diffeomorphic to a given one, while for the case of the ideal fluid the domain is replaced by the class of the isovorticed fields, i.e., by the fields with diffeomorphic vorticities. (The term "dual" above refers to the fact that the domain of diffeomorphic fields is an adjoint orbit in the Lie algebra of all divergence-free vector fields, whereas the isovorticed fields constitute a coadjoint orbit of that algebra; see Chapter I.)

The extremal fields in both of the variational problems coincide ([Arn9]; for the proof see Section II.2). These fields have very peculiar topology (cf. Section II.1). Namely, the extremals are the divergence-free fields that commute with their vorticities. They are either Beltrami flows (i.e., the fields proportional to their own vorticities) or are "integrable" flows whose stream lines fill almost everywhere tori and annuli; see Fig. 9 in Chapter II.

This analysis of topology of the extremal fields leaves little hope that the idealized physical model of the magnetic field relaxation, described above, is legitimate for any somewhat general initial conditions. Indeed, the initial magnetic field **B** can be chosen having no invariant magnetic surfaces. Then the terminal field, if there is one, cannot have invariant tori or annuli and must be a solenoidal field of a very special (Beltrami) type (see [Hen] for the first numerical evidence of chaos in the Beltrami flows). But such fields are too scarce, and one could hardly find a field with the prescribed topology of the magnetic lines amongst them.

It appears that for a correct description of the actual process it is necessary to take into account the magnetic viscosity, which violates the assumption that the field is frozen-in and implies "reconnection" of the magnetic lines. Such a process was not taken care of in our initial system of equations (one has to add the term $\mu \Delta B$ on the right-hand side of the second equation to capture this phenomenon).

Question 1.2. To what extent can one use the extremal fields to study the behavior of the magnetic field **B** at large time scales? What phenomena should appear over the time interval during which the ordinary viscosity succeeds in extinguishing the motion of the fluid, but the magnetic viscosity would not yet extinguish the field **B**?

1.C. Helicity bounds the energy

Let ξ be a divergence-free vector field defined in a simply connected domain $M \subset \mathbb{R}^3$ and tangent to the boundary of M.

Definition 1.3. The *helicity* (or the *Hopf invariant*) of the field ξ in the domain $M \subset \mathbb{R}^3$ is

$$\mathcal{H}(\xi) = \langle \xi, \mathrm{curl}^{-1}\, \xi \rangle = \int_M (\xi, \mathrm{curl}^{-1}\, \xi)\, dV,$$

where (,) is the Euclidean inner product, and $A = \mathrm{curl}^{-1}\, \xi$ is a divergence-free vector potential of the field ξ; i.e., $\nabla \times A = \xi$, $\mathrm{div}\, A = 0$.

The integral is independent of the particular choice of A (which is defined up to addition of the gradient ∇f of a harmonic function, since M is simply connected). Indeed, integrating by parts, one obtains the following expression for the difference

of the helicity values associated to two different choices of A:

$$\int_M (\xi, A_1)\, \mu - \int_M (\xi, A_2)\, \mu = \int_M (\xi, \nabla f)\, \mu = \int_M (f \operatorname{div} \xi)\, \mu + \int_{\partial M} (f \cdot \xi)\, dS = 0,$$

where the last term vanishes, since ξ is tangent to the boundary ∂M. Note that such a field $A = \operatorname{curl}^{-1} \xi$ exists and is defined uniquely in a simply connected M upon specification of the boundary conditions; e.g., A is tangent to the boundary of M (or, more generally, the normal to the boundary ∂M component (A, n) of the vector field A is fixed). If M is not bounded (say, $M = \mathbb{R}^3$), the field ξ is supposed to decay at infinity fast enough to make the integral above converge.

The helicity of a field measures the average linking of the field lines, or their relative winding (see details in Section 1.D below).

Though the idea of helicity goes back to Helmholtz and Kelvin (see [Kel]), its second birth in magnetohydrodynamics is due to Woltjer [Wol] and in ideal hydrodynamics is due to Moffatt [Mof1], who revealed its topological character (see also [Mor2]). The word "helicity" was coined in [Mof1] and has been widely used in fluid mechanics and magnetohydrodynamics since then. We refer to [Mof2, MoT] for nice historical surveys.

The principal feature of this concept is described in the following statement.

Theorem 1.4 (Helicity invariance). *The helicity $\mathcal{H}(\xi)$ is preserved under the action on ξ of a volume-preserving diffeomorphism of M.*

In this sense $\mathcal{H}(\xi)$ is a topological invariant: Though it is defined above with the help of a metric, every volume-preserving diffeomorphism carries a field ξ into a field with the same helicity. We will prove this theorem in a slightly more general setting at the end of this section just by giving a metric-free definition of the invariant. Now we get an immediate and important dividend:

Theorem 1.5 [Arn9]. *For a divergence-free vector field ξ,*

$$E(\xi) \geq C \cdot |\mathcal{H}(\xi)|,$$

where C is a positive constant dependent on the shape and size of the compact domain M.

Proof. The proof is a composition of the Schwarz inequality

$$\mathcal{H}^2(\xi) = \langle \xi, A \rangle^2 \leq \langle \xi, \xi \rangle \langle A, A \rangle$$

and the Poincaré inequality, applied to the vector field A (tangent to the boundary of M if $\partial M \neq \emptyset$):

$$\langle A, A \rangle = \int_M (A, A)\mu \leq \frac{1}{C^2} \int_M (\xi, \xi)\mu = \frac{1}{C^2} \langle \xi, \xi \rangle$$

for $A = \text{curl}^{-1} \xi$, $E(\xi) = \langle \xi, \xi \rangle$. $\square$

Various applications of this theorem can be found in [MoT, L-A, CDG].

Remark 1.6. The inverse (nonlocal) operator curl^{-1} sends the space of divergence-free vector fields (tangent to the boundary) on a simply connected manifold onto itself. This operator is symmetric, and its spectrum accumulates at zero on both sides. The restriction of the operator $-\text{curl}^2$ to the space of the divergence-free vector fields is called the Laplace–Beltrami operator on the divergence-free fields. Its components in the Euclidean $\mathbb{R}^3$ case are the Laplacians of the field components. Its spectrum is a sequence of real numbers divergent to $-\infty$.

This Laplacian $-\text{curl}^2$ differs by the sign from the Laplace operator of topologists $d\delta + \delta d$ (see Sections 1.D and V.3.B below) restricted to the space of closed two-forms. Here a divergence-free vector field ξ on a Riemannian manifold is regarded as the corresponding closed 2-form $i_\xi \mu$.

Corollary 1.7. *The eigenfield of the operator* curl^{-1} *corresponding to the eigenvalue* λ *of the largest absolute value has minimal energy within the class of divergence-free fields obtained from this eigenfield by the action of volume-preserving diffeomorphisms.*

Indeed, for any field ξ the energy $E(\xi)$ can be minorized as follows:

$$E(\xi) = \langle \xi, \xi \rangle \geq \frac{1}{\lambda} \langle \text{curl}^{-1} \xi, \xi \rangle,$$

and the inequality becomes the equality for the eigenfield with the eigenvalue λ. In general, the constant C of the preceding theorem can be taken equal to $|\lambda|$.

Remark 1.8. The theorems above, as well as many results below, hold for the more general case of manifolds M whose first homology group vanishes: $H_1(M, \mathbb{R}) = 0$.

This statement also holds for an arbitrary closed three-dimensional Riemannian manifold if one confines oneself to divergence-free fields that are "null-homologous" (i.e., have a single-valued divergence-free potential).

Example 1.9. The standard *Hopf vector field* on

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | \sum_{i=1}^{4} x_i^2 = 1\}$$

is defined by

$$\mathbf{v}(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3).$$

It corresponds to the maximal eigenvalue (=1/2) of the curl^{-1} operator on S^3 with the canonical induced metric and the orientation given by the inner normal. The

trajectories of this field are the great circles along which $S^3 \subset \mathbb{C}^2$ intersects the complex lines $\mathbb{C}^1 \subset \mathbb{C}^2$ (see Fig. 21 for **v**-orbits under stereographic projection $S^3 \to \mathbb{R}^3$). These trajectories are pairwise linked. The Hopf field on S^3 has minimal energy among all the fields diffeomorphic to it, i.e., obtainable from it by the action of a volume-preserving diffeomorphism.
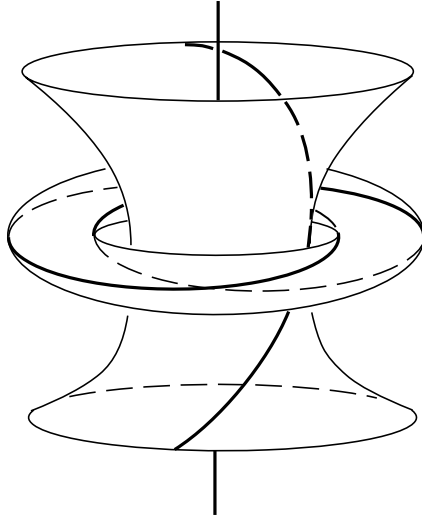
FIGURE 21.   Trajectories of the Hopf field in $\mathbb{R}^3$ (the stereographic projection from S^3). One circle becomes the vertical axis. Every two orbits are linked.

1.D. Helicity of fields on manifolds

We consider here an *ad hoc* definition of the helicity integral on manifolds [Arn9], establish its simplest properties (in particular, the topological invariance), and identify the result with Definition 1.3 above. An interesting topological meaning of the invariant will be discussed in the next two sections.

Let M be a three-dimensional manifold that is closed (compact, without boundary), oriented, and connected, and let μ be a volume element (i.e., a nonvanishing differential 3-form defining the correct orientation) on M. Notice that we fix a volume element on M, but we do not select any Riemannian metric.

Definition 1.10. Every vector field ξ on M generates a differential 2-form ω_ξ according to the formula

$$\omega_\xi(\eta, \zeta) = \mu(\xi, \eta, \zeta)$$

for any vector fields η and ζ. The correspondence $\xi \mapsto \omega_\xi = i_\xi \mu$ is an isomorphism of the linear spaces of fields and 2-forms. The differential of ω_ξ, being a

3-form, can be expressed via the volume form as

$$d\omega_\xi = \varphi \cdot \mu,$$

where $\varphi : M \to \mathbb{R}$ is a smooth function. The function φ is called the *divergence* of the field ξ: $\varphi = \operatorname{div} \xi$. The velocity field of a flow that preserves the volume element on M is divergence free, and conversely every field with vanishing divergence on M is the velocity field of an incompressible flow.

Remark 1.11. The origin of divergence is explained by the homotopy formula for the Lie derivative $L_\xi = i_\xi d + d i_\xi$. The Lie derivative L_ξ is the derivative of any differential form f along the vector field ξ, defined as the derivative of the form $g^{t*} f$ transported by the flow g^t of the vector field ξ, evaluated at the initial moment $t = 0$: $L_\xi f = \frac{d}{dt}|_{t=0}(g^{t*} f)$. The operation i_ξ is the substitution of the vector field ξ in the differential form as the first argument, and d is the (exterior) derivative. Applied to the form μ it gives $L_\xi \mu = i_\xi d\mu + d i_\xi \mu = d\omega_\xi = \varphi\mu$. Thus the function φ is the coefficient of stretching (or divergence) of the volume form by the field ξ.

Definition 1.12. A divergence-free vector field ξ on M is said to be *null-homologous* if the 2-form ω_ξ corresponding to it is the differential of a globally defined 1-form α on M:

$$\omega_\xi = d\alpha.$$

The 1-form α will be called a *form-potential*. A field is null-homologous if and only if its flux across every closed surface is zero. In the case of a simply connected closed M every divergence-free vector field is null-homologous.

Remark 1.13. If M is endowed with a Riemannian metric $(\ ,\ )$ then the 1-form α can be identified with the vector field A for which

$$\alpha(\eta) = (A, \eta) \quad \text{for every field } \eta.$$

Here $\xi = \operatorname{curl} A$ (in the Euclidean case $\xi = \nabla \times A$), and the vector field A is called the *vector-potential* of ξ. We would like to make a point, however, that the forms ω and α (in contrast to the field A) do not depend on the Riemannian metric but rely only on the choice of the volume element μ.

Definition 1.14. The *helicity (or Hopf invariant)* $\mathcal{H}(\xi)$ of a null-homologous field ξ on a three-dimensional manifold M (possibly with boundary) equipped with a volume element μ is the integral of the wedge product of the form ω_ξ and its form potential:

$$\mathcal{H}(\xi) = \int_M \alpha \wedge d\alpha = \int_M d\alpha \wedge \alpha, \qquad \text{where} \quad d\alpha = \omega_\xi.$$

Theorem 1.15. *This definition is consistent; i.e., the value of $\mathcal{H}$ does not depend on the particular choice of the form-potential α, but only on the field ξ:*

    (*i*) *for a manifold M without boundary, or*

   (*ii*) *for a simply connected manifold M with boundary, provided that the field*
       ξ *tangent to* ∂M.

Proof. (*i*) First assume that M is without boundary. If $\beta = \alpha + \theta$ is another form potential for the same 2-form ω_ξ, then $d\theta = 0$, and therefore

$$\int_M \alpha \wedge d\alpha - \beta \wedge d\beta = \int_M \theta \wedge d\alpha = \int_M d(\theta \wedge \alpha) = \int_{\partial M = \emptyset} \theta \wedge \alpha = 0.$$

(*ii*) Now $\partial M \neq \emptyset$. In the simply connected case, a variation θ of the form-potential is exact ($\theta = df$ for some function f on M), and the variation of $\mathcal{H}$ is given by

$$\int_M \theta \wedge d\alpha = \int_M df \wedge d\alpha = \int_M d(f \wedge d\alpha) = \int_{\partial M} f \wedge d\alpha = 0,$$

where $d\alpha$ vanishes on ∂M due to the condition $\xi || \partial M$. $\square$

Remark 1.16. In the presence of a Riemannian metric on M the helicity can be expressed as

$$\mathcal{H}(\xi) = \int_M \alpha \wedge \omega_\xi = \int_M \alpha \wedge i_\xi \mu = \int_M \alpha(\xi) \wedge \mu = \int_M (A, \xi)\mu = \langle \mathrm{curl}^{-1} \xi, \xi \rangle,$$

where A is any vector-potential of ξ. (The shift of the substitution operator from μ to α is due to the fact that i_ξ is the (inner) differentiation: $i_\xi(\alpha \wedge \mu) = i_\xi \alpha \wedge \mu - \alpha \wedge i_\xi \mu$.) Therefore, consistent with Definition 1.3, $\mathcal{H}$ is the inner product of the field with its vector potential.

The above coordinate-free approach can be summarized in the following

Corollary 1.17. *The helicity of a null-homologous vector field ξ is preserved under the action of an arbitrary volume-preserving diffeomorphism of M. For a simply connected manifold M with boundary, the helicity of a divergence-free vector field tangent to the boundary does not change under the action of all volume-preserving diffeomorphisms of M that carry the boundary ∂M to itself.*

In particular, on a Riemannian manifold the inner product of a divergence-free field and its vector potential is preserved when the field is acted on by a volume-preserving diffeomorphism.

Proof. The invariance of $\mathcal{H}$ under diffeomorphisms that preserve the volume element follows from the fact that $\mathcal{H}$ can be defined by using no structures other than the smooth structure of M and the volume element μ. $\square$

This observation constitutes the proof of the Helicity Invariance Theorem.

Example 1.18 (= 1.9′). The helicity of the Hopf vector field on $S^3 \subset \mathbb{R}^4$ (defined in Example 1.9, Fig. 21) is $\pi^2/2$. Indeed,

$$\mathcal{H}(\mathbf{v}) = \int_{S^3} (\mathbf{v}, \mathrm{curl}^{-1} \mathbf{v})\mu = \frac{1}{2} \int_{S^3} (\mathbf{v}, \mathbf{v})\mu = \frac{1}{2} \int_{S^3} \mu = \frac{\mathrm{vol}(S^3)}{2} = \frac{2\pi^2}{2} = \pi^2,$$

since the eigenvalue of the curl^{-1} operator on S^3 is equal to $-\frac{1}{2}$, and the volume of S^3 is $2\pi^2$.

Example 1.19. With any smooth map $\pi : S^3 \to S^2$ one can associate the following integer number, called the *Hopf invariant* of π. Fix on the sphere S^2 an arbitrary area form ν normalized by the condition $\mathrm{area}(S^2) := \int_{S^2} \nu = 1$. Such a form is closed on the sphere S^2, and hence its pullback $\pi^*\nu$ is exact on S^3 (since $H^2(S^3) = 0$). That is, there exists a 1-form α such that $d\alpha = \pi^*\nu$. Then the Hopf invariant of π is

$$\mathcal{H}(\pi) = \int_{S^3} \alpha \wedge \pi^*\nu.$$

Proposition 1.20. $\mathcal{H}(\pi)$ *is an integer.*

Proof hint: Choose the form ν to be a δ-type form on S^2 supported at one point only. Compare the result with the topological definition of the Hopf invariant below. □

Given a volume form on S^3, the number $\mathcal{H}(\pi)$ is the helicity of the divergence-free vector field ξ defined by the condition $i_\xi \mu = \pi^*\nu$. The orbits of this field are closed, being the preimages of points of S^2 under the mapping π. The above definition of the helicity is a generalization of the Hopf invariant to the case where an exact 2-form on S^3 (or on M^3) is not necessarily a pullback for any map π.

An equivalent (topological) definition of the Hopf invariant for a map $S^3 \to S^2$ is the linking number in S^3 of the preimages of two generic points in S^2 (Fig. 22). The equivalence of the topological and integral definitions plays a key role in what follows in this chapter.

Theorem 1.5 claims that for a map $\pi : S^3 \to S^2$ with nonzero Hopf invariant $\mathcal{H}(\pi)$, (a multiple of) the absolute value of this invariant bounds below the energy of the corresponding vector field. The latter field is directed along the fibers of the map π. The length of the vectors is defined by the volume form on S^3 and the pullback of the S^2 area element.

Remark 1.21. L.D. Faddeev proposed another but relevant variational problem for the mappings π from $\mathbb{R}^3$ to S^2. Consider the functional on such mappings that is a (weighted) sum of two terms. The first term is the Dirichlet integral (of the squared derivative) of the map π. The second term is the energy of the
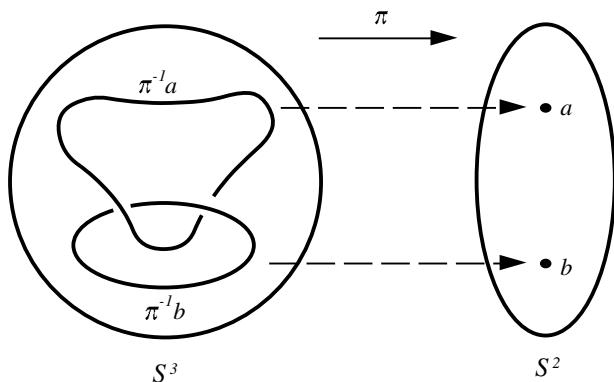
FIGURE 22.   Hopf invariant for a map $S^3 \to S^2$.

corresponding vector field directed along the fibers of the map. Then this functional is bounded below by (a multiple of) $|\mathcal{H}(\pi)|^{3/4}$, where $\mathcal{H}(\pi)$ is the Hopf invariant of the map $\pi : \mathbb{R}^3 \to S^2$ [V-K]. The proof uses a version of the Sobolev inequality [Sob1]; cf. Theorem 5.3 below and its proof, which employs the same inequality.

Furthermore, some recent computer experiments for the relaxation process of an initial mapping with nonzero Hopf invariant exhibit the following phenomenon. In the equivariant case (of S^1 acting by rotations on R^3 and S^2), one observes an "energy gap" over the poles, where the rotation axis intersects the sphere. It would be very interesting to explain this singularity structure.

The addition of the Dirichlet integral to the energy is similar to the addition of the Lagrange multiplier in the problem of energy minimization. We could start with the action of all diffeomorphisms, and then consider the conditional minimum for the action of only volume-preserving ones.

Remark 1.22. The Hopf invariant equips the Lie algebra of divergence-free vector fields on a closed simply connected three-dimensional manifold with a *bilinear form*:

$$\mathcal{H}(\xi, \eta) = \langle \xi, \operatorname{curl}^{-1} \eta \rangle,$$

where $\operatorname{curl}^{-1} \eta$ is a vector-potential of the field η.

This form is invariant with respect to the natural action of volume-preserving diffeomorphisms on vector fields (i.e., with respect to the adjoint representation of the group $S\operatorname{Diff}(M)$ in its Lie algebra; see Chapter I). Moreover, the form $\mathcal{H}$ is *symmetric*, since

$$\mathcal{H}(\xi, \eta) = \int_M i_\xi \mu \wedge d^{-1}(i_\eta \mu) = \int_M d^{-1}(i_\xi \mu) \wedge i_\eta \mu$$
$$= \int_M i_\eta \mu \wedge d^{-1}(i_\xi \mu) = \mathcal{H}(\eta, \xi).$$

The positive and negative subspaces of the form $\mathcal{H}$ are both infinite-dimensional; see [Arn9, Smo1]. Thus $\mathcal{H}$ generates a *bi-invariant pseudo-Euclidean* (indefinite) metric on the corresponding group $S\,\mathrm{Diff}\,(M)$. For the case of a non-simply connected M one has to confine oneself to the subalgebra of all null-homologous vector fields within the Lie algebra of all divergence-free vector fields on M (see Section IV.8.D for more detail).

In this case one may also hope to define the generalized Hopf invariants with values in some modules over the fundamental group, but this way has not yet been duly explored.

§2. Topological obstructions to energy relaxation

2.A. *Model example: Two linked flux tubes*

The helicity obstruction to the energy relaxation is clearly seen in the example of a magnetic field confined to two linked solitori; Fig. 23a,b. Assume that the field vanishes outside those tubes and the field trajectories are all closed and oriented along the tube axes inside.

To minimize the energy of a vector field with closed orbits by acting on the field by a volume-preserving diffeomorphism, one has to shorten the length of most trajectories. (Indeed, the orbit periods are preserved under the diffeomorphism action; therefore, a reduction of the orbits' lengths shrinks the velocity vectors along the orbits.) In turn, the shortening of the trajectories implies a fattening of the solitori (since the acting diffeomorphisms are volume-preserving).

(a)   (b)
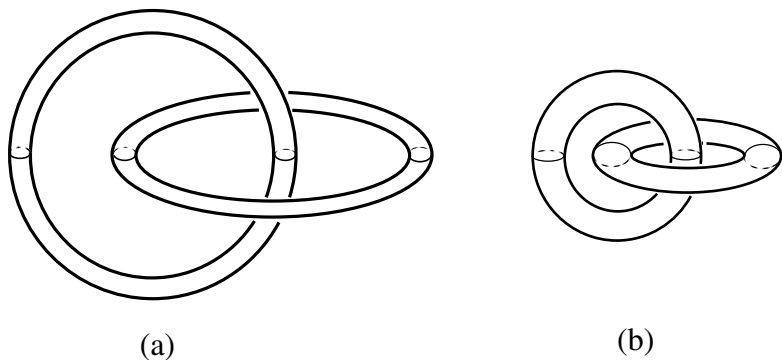
FIGURE 23.   (a) A magnetic field is confined to two linked solitori. (b) Relaxation fattens the tori and shrinks the field orbits.

For a linked configuration, as in Fig. 23b, the solitori prevent each other from endless fattening and therefore from further shrinking of the orbits. Therefore, heuristically, in the volume-preserving relaxation process the magnetic energy of the field supported on a pair of linked tubes is bounded from below and cannot attain too small values [Sakh].

Below we show that the helicity of a field measures the rate of the mutual winding (or "helix-likeness") of the field trajectories around each other. To visualize this notion (and the paradigm "helicity bounds energy" of the preceding section), we first concentrate on the degenerate situation above (see [Mof1]).

Let a magnetic (that is, divergence-free) field ξ be identically zero except in two narrow linked flux tubes whose axes are closed curves C_1 and C_2. The magnetic fluxes of the field in the tubes are Q_1 and Q_2 (Fig. 24).
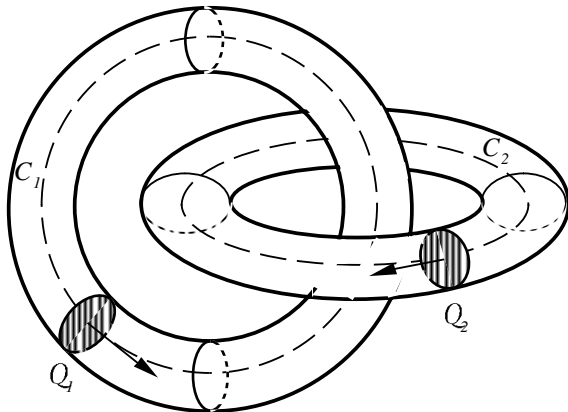
FIGURE 24. C_1, C_2 are axes of the tubes; Q_1, Q_2 are the corresponding fluxes.

Suppose further that there is no net twist within each tube or, more precisely, that the field trajectories foliate each of the tubes into pairwise unlinked circles.

Lemma 2.1. *The helicity invariant of such a field is given by*

$$(2.1) \qquad \mathcal{H}(\xi) = 2\,lk(C_1, C_2) \cdot Q_1 \cdot Q_2,$$

where $lk(C_1, C_2)$ is the linking number of C_1 and C_2.

Definition 2.2. The (Gauss) *linking number* $lk(\Gamma_1, \Gamma_2)$ of two oriented closed curves Γ_1, Γ_2 in $\mathbb{R}^3$ is the signed number of the intersection points of one curve with an arbitrary (oriented) surface bounded by the other curve (Fig. 25). The sign of each intersection point is defined by the orientation of the 3-frame that is formed at this point by the velocity vector of the curve and by the 2-frame orienting the surface.

The linking number of curves is symmetric: $lk(\Gamma_1, \Gamma_2) = lk(\Gamma_2, \Gamma_1)$.

Proof of Lemma. The helicity volume integral $\mathcal{H}(\xi) = \langle \mathrm{curl}^{-1}\xi, \xi \rangle = \int (A, \xi)\mu$ over the tubes (here $A = \mathrm{curl}^{-1}\xi$) descends to the sum of the cor-
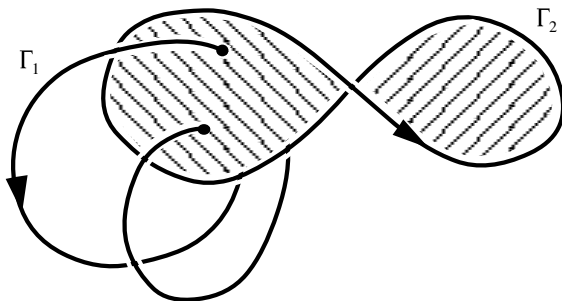
FIGURE 25.   The linking number of Γ_1 and Γ_2 is the signed number of intersections of Γ_1 with a surface bounded by Γ_2.

responding line integrals:

$$\mathcal{H}(\xi) = Q_1 \int_{C_1} (A,\, dC_1) + Q_2 \int_{C_2} (A,\, dC_2).$$

Indeed, the volume element μ in each tube is the product of the line element dC_i and the area element dS_i of the tube cross section. In turn, the integral of the $\xi\, dS_i$ over the corresponding cross section is the flux Q_i. Hence,

$$\int_{i\text{th tube}} (A, \xi)\, dS_i\, dC_i = \int_{C_i} \int_{S_i} (A, (\xi\, dS_i)\, dC_i) = Q_i \int_{C_i} (A,\, dC_i).$$

Furthermore, the circulation $\int_{C_1} (A,\, dC_1)$ of the field A over the curve C_1 is the full flux of curl $A = \xi$ through a surface bounded by the axis curve C_1. The latter flux is equal to $Q_2 \cdot lk(C_1, C_2)$: Every crossing of the surface by the second tube contributes to the signed amount of Q_2 into the full flux. Note that the first tube itself does not contribute into that flux through its axis C_1, due to the assumption on the net twist within the tubes.

The same argument applied to the second circulation integral doubles the result:
$\mathcal{H}(\xi) = 2\, lk(C_1, C_2) \cdot Q_1 \cdot Q_2$. $\square$

A generalization of this example to the case of an arbitrary divergence-free vector field ξ is described in Section 4.

2.B. *Energy lower bound for nontrivial linking*

The linking number is a rather rough invariant of a linkage. The signed number entering the definition of *lk* can turn out to be zero for configurations of curves linked in an essential way (see, e.g., the so-called Whitehead link in Fig. 26a). However, the heuristic observation of the beginning of Section 2.A for the energy bound still holds.

The heuristics above are supported by the following result of M. Freedman [Fr1]: Any essential linking between circular packets of ξ-integral curves implies a lower bound to E.
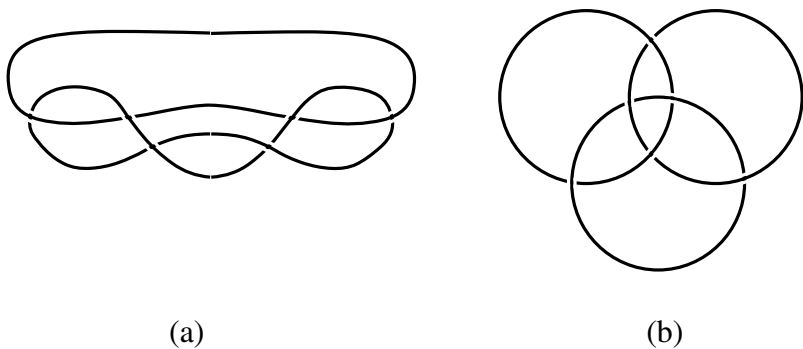
FIGURE 26.   Nontrivial links with vanishing pairwise linking numbers. (a) Whitehead link. (b) Borromean rings.

Definitions 2.3. A link *L*, i.e., a smooth embedding of *n* circles into a 3-dimensional manifold, is *trivial* if it bounds *n* smoothly and disjointly embedded disks. Otherwise, the link is called *essential.*

A vector field ξ on *M* is said to be *modeled on L* if there is a ξ-invariant tubular neighborhood of $L \subset M$ foliated by integral curves of ξ that is diffeomorphic to $\bigcup_{i=1}^{n} D_i^2 \times S^1$ foliated by circles {point} $\times S^1$ (here D^2 is a 2-dimensional disk).

Theorem 2.4 [Fr1]. *If ξ is a divergence-free vector field on a closed 3-manifold M that is modeled on an essential link, (or knot) L, then there is a positive lower bound to the energy of fields obtained from ξ by the action of volume-preserving diffeomorphisms of M.*

Under the additional assumption on a field to be strongly modeled on a link, the lower energy bound for a field in $\mathbb{R}^3$ was obtained in [FH1] explicitly. A divergence-free field ξ is *strongly modeled on L* if there is a volume-preserving embedding that carries the field $\frac{\partial}{\partial\theta}$ directed along the circles in $\bigcup_{i=1}^{n}(D^2 \times S^1)_i$ into ξ within a tubular neighborhood of *L*. The neighborhood consists of several solid tori of equal volume, which we denote by *V*.

Theorem 2.5 [FH1]. *The energy of a vector field ξ strongly modeled on an essential link L in $\mathbb{R}^3$ satisfies the inequality*

$$E(\xi) > \left(\frac{\sqrt{6/125}}{\pi^2} \right)^{4/3} \cdot V^{5/3} \approx 0.00624 \quad V^{5/3}.$$

Note that given any link, one may construct a field modeled (and even strongly modeled) on it. The exponent 5/3 has the following origin. The Euclidean dilation with a factor *l* multiplies the image field by *l* and the volume element by l^3. Thus

the total energy gains the factor l^5, while the volume is multiplied by the factor l^3. Hence, the ratio $E/V^{5/3}$ is purely geometrical and independent of scaling in the Euclidean case.

Remark 2.6. Theorem 2.5 suggests the following construction of a set of invariants of topological or smooth 3-manifolds. The invariants are parametrized by the isotopy classes of knots and links in the manifold. They might also be regarded as the invariants of embeddings of 1-dimensional manifolds into 3-dimensional ones.

Consider the ratio $E/V^{5/3}$ for a vector field strongly modeled on the knot or on the link of a given isotopy class in a Riemannian manifold. Take the infimum over all such fields and over all the Riemannian metrics. The resulting number is an invariant of the smooth (perhaps, even topological) isotopy class of the pair (link, 3-manifold).

Further, one might take the infimum over all the compact 3-manifolds for a homotopically trivial link to get an invariant of the classical link or knot. (Is this infimum equal to the infimum of the above ratio for Euclidean 3-space or for the 3-sphere? Is the supremum over all the 3-manifolds finite?)

One might also start with a compact Riemannian manifold of volume 1 and with a link of k solid tori of volume V each. If kV is smaller than 1, the infimum of $E/V^{5/3}$ over the metrics of total volume 1 is a function of V, which is still an invariant of the embedding. We do not know whether these invariants are nontrivial, i.e., whether they distinguish any 3-manifolds or embeddings (cf. Remark 6.7).

Freedman and He have informed us that Theorem 2.5 can be generalized to arbitrary Riemannian manifolds. The limit of the coefficient $C(V)$ for small volumes V is the same constant as in the Euclidean case $C = \left(\sqrt{6/125}/\pi^2\right)^{4/3}$ given by Theorem 2.5.

The strongly modeled fields have very simple behavior near the link and are far from being generic within divergence-free vector fields. It would be of interest to completely get rid of the condition on a special tubular neighborhood.

Problem 2.7. Is there an energy lower bound for a field having a set of closed trajectories forming an essential link on a Riemannian manifold (without an assumption on a neighborhood of closed orbits)?

Remark 2.8. The strongest result in this direction was obtained in [FH2] (see Section 5), where the condition on a field to be modeled on a link was weakened to the requirement for a field in $\mathbb{R}^3$ to have invariant tori confining the link components. Such fields form an ample set near the integrable divergence-free flows. This follows from the KAM theory of Hamiltonian perturbations of integrable Hamiltonian systems.

In particular, if a closed field orbit is *elliptic* (and generic), i.e., its Poincaré map has two eigenvalues of modulus 1, then this orbit is confined to a set of nested tori invariant under the field (see, e.g., [AKN]). Thus, *every such orbit forming*

an essential knot provides the lower bound for the energy of the corresponding field. Indeed, the energy of any of the invariant solid tori confining this knotted orbit cannot diminish to zero, according to [FH2]. One can argue that a vector field with a knotted *hyperbolic* closed orbit (whose Poincaré map has real eigenvalues of the modulus different from 1) may not have a positive lower bound for the energy (cf. the next section).

Remark 2.9. The different estimates for the magnetic energy, should magnetic solid tori form a trivial or nontrivial link, have a striking counterpart in the theory of Brownian motion.

Let K be a knot in S^3, and $\{z(t) \mid t > 0\}$ the standard Brownian motion on S^3 starting at some point $O \notin K$ at a distance $d(O, K) = \tau > 0$ from K. If K is unknotted, then there exists almost surely a sequence $t_1 < t_2 < \cdots$ such that $t_n \to \infty$ and for which $d(z(t_n), O) \leq \tau/2$. Furthermore, the loop that we obtain by following the Brownian path up to $z(t_n)$ and then joining $z(t_n)$ to O by a short path $\Delta(z(t_n), O)$ is homotopic to O in $S^3 \setminus K$ [Var]. In other words, (almost surely) the Brownian path returns close to its starting point untangled with respect to K, and it does this infinitely many times.

The exact opposite happens when K is knotted: There almost surely exists a $T > 0$ such that whenever the distance $d(z(t), O)$ is small enough, $d(z(t), O) \leq \tau/2$ and $t > T$, the homotopy class of the above loop is not trivial [Var]. In this sense the Brownian motion can tell whether K is an essential knot or not. Heuristically, this means that the Brownian motion distinguishes the existence of a hyperbolic metric on the universal covering to $S^3 \setminus K$ (see Thurston's theorem on the hyperbolic structure on the complement to a nontrivial knot or link [Th2]). More details on Brownian motion in the presence of knots, as well as on various topological problems related to polymers can be found in [KhV].

§3. Sakharov–Zeldovich minimization problem

Assume now that a divergence-free field has a trivial topology in that all field trajectories are closed and pairwise unlinked. An example of such a field is the rotation field in a 3-dimensional ball (Fig. 27). The energy lower bounds considered in Section 2 are valid for essential links and are not applicable here. On the contrary, in this case the field energy can be reduced almost to zero by a keen choice of volume-preserving diffeomorphisms [Zel2, Sakh, Arn9, Fr2].

Theorem 3.1. *The energy of the rotation field in a 3-dimensional ball can be made arbitrarily close to zero by the action of a suitable diffeomorphism that preserves volumes and fixes the points in a neighborhood of the ball boundary.*

Remark 3.2. This result, formulated by A. Sakharov and Ya. Zeldovich [Sakh, Zel2], is based on the following reasoning. Divide the whole ball into a number of thin solid tori (bagels) formed by the orbits of the field and into a remainder of small volume. Then deform each solid torus (preserving its volume) such that it becomes fat and small, with the hole decreasing almost to zero. (Such deformations must
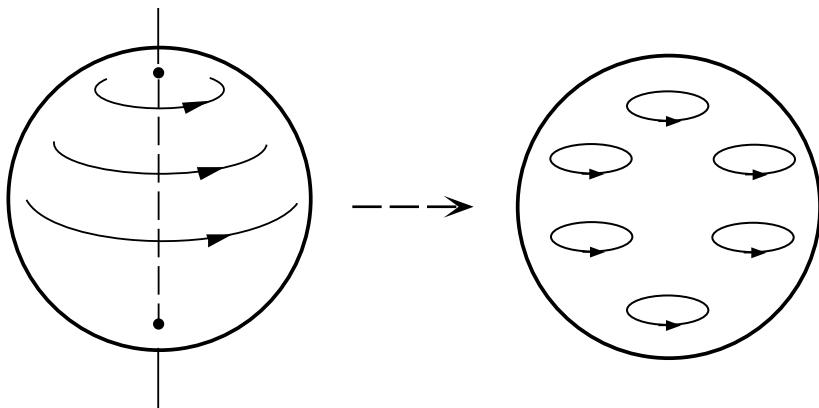
FIGURE 27.   A rotation field in a 3-dimensional ball can dissipate its energy almost completely.

violate the axial symmetry of the field, since any axisymmetric diffeomorphism sends the rotation field to itself and hence preserves the total energy.) Now the field energy in the solid tori is decreased (since the field lines are shortened). The whole construction can be carried out in such a way that the field energy in the remaining small volume is not increased by too much. As a result, the total energy remains arbitrarily small.

This consideration was placed on a rigorous foundation by M. Freedman. We outline the main ideas of his proof below.

Let B^3 be a ball in three-dimensional Euclidean space and ξ the vector field generated by infinitesimal rotation about the vertical axis. The trajectories of this field are horizontal pairwise unlinked circles (and their limits, the points on the vertical axis).

Theorem 3.3 [Fr2]. *There exists a family of volume-preserving diffeomorphisms $\varphi_t : B^3 \to B^3$, $1 \le t \le \infty$, such that it starts at the identity diffeomorphism ($\varphi_1 = \mathrm{Id}$), it is steady on the boundary ($\varphi_t\big|_{\partial B^3} = \mathrm{Id}$) for all t, and the family of the transformed vector fields $\xi_t := \varphi_{t*}\xi$ (being the image of the rotation field ξ under the φ_t-action) fulfills the following conditions as $t \to \infty$:*

(1) *the energy of the field ξ_t decays as $E(\xi_t) := \|\xi_t\|_{L^2}^2 = \mathcal{O}(1/t)$,*
(2) *the supremum norm is unbounded: $\|\xi_t\|_{L^\infty} = \mathcal{O}(t)$, yet*
(3) *for all k, $p < \infty$ the Sobolev norms decay: $\|\xi_t\|_{L^{k,p}} \to 0$ (here the norm $\|\eta\|_{L^{k,p}}$ is the L^p-norm in the space of η's derivatives of orders $0, \dots, k$).*

Remark 3.4 [Fr2]. For this family of diffeomorphisms, the limit of $\xi_t = \varphi_{t*}\xi$ at infinity $t \to \infty$ does not exist, but for large t the regions of large norm $\|\xi_t\|$ constitute a "topological froth" $\mathcal{F}_t$ with trivial relative topology. The froth $\mathcal{F}_t$ is

a "time-fractal" (the facet size drops abruptly in a sequence of catastrophes as t increases) and becomes dense as $t \to \infty$.

Proof sketch. The following lemma is a modification of Moser's result [Mos1] on the existence of volume-preserving diffeomorphisms between diffeomorphic manifolds of equal volume.

Lemma 3.5. *Let D and D' be domains of equal volume in $\mathbb{R}^m$ and $f : D \to D'$ a diffeomorphism. Then f is isotopic to a volume-preserving diffeomorphism f_0 between the domains.*

Moreover, if f preserves orientation and a function ρ is the "excess density" $\rho = 1 - \det(f_*)$, then there exist constants $C_{k,p}$ depending only on the domain D such that

$$\|f - f_0\|_{L^{k+1,p}} \leq C_{k,p}\|\rho\|_{L^{k,p}} \qquad \textit{for any} \quad k, p < \infty.$$

Proof of Lemma 3.5. Pull back the D'-volume form $\mu_{D'}$ to D. The density function ρ manifests the excess of the volume $f^*(\mu_{D'})$ over μ_D. The mean value of ρ is zero due to the volume equality condition.

Let ψ be a solution of the Neumann problem on D for ρ; i.e., $\Delta\psi = \rho$ on D and $\frac{\partial}{\partial n}\psi = 0$ on the boundary ∂D (where $\partial/\partial n$ indicates differentiation in the direction of the exterior normals; see Lemma 3.7 below on solvability of the Neumann problem).

Rewrite this system in the form $\operatorname{div}(\nabla\psi) = \rho$, $\nabla\psi \parallel \partial D$. Then the gradient vector field $\nabla\psi$ is tangent to the boundary ∂D and defines infinitesimally an isotopy of D moving the volume element μ_D into $f^*(\mu_{D'})$. The isotopy itself is now the phase flow of the dynamical system on D defined by the instant field $\nabla\psi$.

Finally, the required estimate is a consequence of the inequality $|\lambda_1| \cdot \|\psi\|_{L^2} \leq \|\rho\|_{L^2}$, where λ_1 is the closest to 0 (from the left) eigenvalue of the Neumann problem. Taking the gradient $\nabla\psi$, we lose one order in the Sobolev norm. $\square$

Remark 3.6. For application to the case where D is a spherical shell, note that the constants $C_{k,p}$ may be chosen independent of the thickness. It follows from the fact that the closest to 0 eigenvalue λ_1 of the Neumann problem on the shell tends to the smallest Laplace–Beltrami eigenvalue on the sphere S^2 as the shell thickness goes to zero.

Indeed, the eigenvalues of the Laplace operator on such a shell are sums of those on the sphere and of the eigenvalues of the radial component

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

of the Laplacian. One immediately sees that all but the first eigenfunctions of the latter operator with the Neumann boundary conditions highly oscillate on a short segment. Hence, all but the first corresponding eigenvalues tend to infinity, while the first one goes to zero as the segment shrinks to a point. This very first eigenvalue

is the only eigenvalue that contributes to the eigenvalue λ_1 of the Neumann problem on the shell, and its contribution vanishes as the shell thickness goes to zero.

Lemma 3.7. *The Neumann problem $\Delta\psi = \rho$ on D and $\frac{\partial}{\partial n}\psi = 0$ on the boundary ∂D has solution for any function ρ with zero mean (i.e., for ρ that is L^2-orthogonal to constants on D).*

Proof of Lemma 3.7. The image of an operator is the orthogonal complement to the kernel of the corresponding coadjoint operator. To apply it to the Neumann operator we first find the set of functions h orthogonal to all $\Delta\psi$ with $\frac{\partial}{\partial n}\psi = 0$:

$$0 = \int_D (\Delta\psi)h = -\int_D \nabla\psi\nabla h + \int_{\partial D}(\frac{\partial}{\partial n}\psi)h = \int_D \psi\Delta h - \int_{\partial D}\psi(\frac{\partial}{\partial n}h).$$

Taking as test functions those ψ's that vanish on the boundary ∂D, we obtain that $\Delta h = 0$. Then for a generic ψ, the boundary term is equal to zero, and hence $\frac{\partial}{\partial n}h = 0$ on ∂D. Thus only the constant functions g are orthogonal to the image of the Neumann operator $\Delta\psi$ (with the boundary condition $\frac{\partial}{\partial n}\psi = 0$), and any function orthogonal to constants is in the image of this operator. $\square$

Main construction. We first cut the ball B in two parts by splitting out a spherical shell Sh of thickness s from a subball B_s (Fig. 28). We will fix s later. The internal subball can be stretched in the vertical direction and squeezed into a thin "snake" by a volume-preserving diffeomorphism.
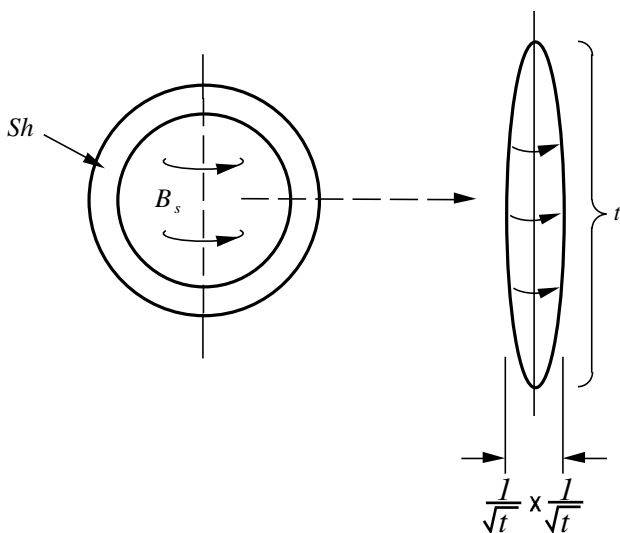
FIGURE 28.   Stretching a subball into a snake reduces its energy.

Such a stretching transformation shrinks all the ξ-orbits (located in the horizontal planes in the internal subball B_s) by an arbitrarily large prescribed factor, and hence it reduces the field energy in the (transformed) subball to an arbitrarily small positive level.

Then we put the snake into the original ball, keeping the volume preserved. Allow the composition map of the subball B_s into a snake inside the ball B to be accompanied with a map of the shell Sh into the snake complement. One may do it first without control of the volume elements but providing smoothness of the transformation $(B_s \cup Sh) \rightarrow B$ (see Fig. 29). Then the accompanying map of the shell Sh can be made volume-preserving by applying the isotopy of Lemma 3.5.
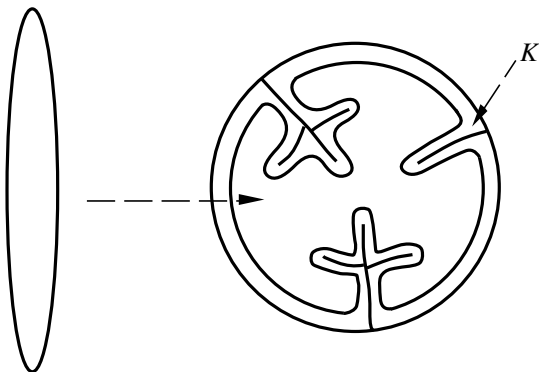
FIGURE 29.   The complement of the snake in the ball is a neighborhood of a 2-complex K.

The total energy of the field ξ after the diffeomorphism action is composed by the energy in the subball image and in the shell image $E = E_{\text{subball}} + E_{\text{shell}}$. The stretching procedure above allows one to handle the first term completely: Given positive ε, the energy E_{subball} can be suppressed to the level $E_{\text{subball}} \approx \varepsilon$ by considering an appropriately long snake. The embedding of the snake into the original ball does not essentially increase its energy, since the bending occurs in the directions orthogonal to the trajectories of the magnetic field, and hence it does not stretch the vectors.

Now we have to estimate the field energy in the shell image. Note that the image is concentrated near a 2-complex K "complementary" to the snake in B. Using Lemma 3.5 and Remark 3.6 in order to control the maximal stretching of orbits in the shell, it is sufficient to provide boundedness of the stretching of the volume element for an arbitrarily thin shell. The latter is achieved by considering a one-parameter family of diffeomorphisms (plotted in Fig. 30):

   (a) first expand a thin shell (of thickness s) to that of a fixed thickness,
   (b) then map it to a neighborhood of K defined by a given snake embedding,
   (c) and finally, squeeze this neighborhood to K.

The energy E_{shell} tends to zero as the thickness $s \rightarrow 0$, since the energy integrand is bounded independently of s, while the volume of the integration domain, the
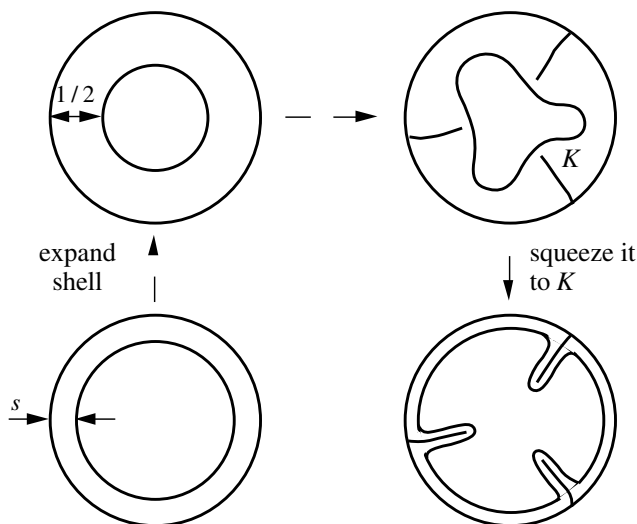
FIGURE 30.   Family of maps of a shell into a neighborhood of K.

shell volume, goes to zero. Thus, having chosen s sufficiently small, one can obtain $E_{\text{shell}} \approx \varepsilon$.

Scale estimate. To organize the family φ_t of diffeomorphisms, we will define the initial stretching of the subball into a snake of length t. Then the area of every horizontal section is squeezed by the factor of t, and vectors themselves are squeezed by the factor of $\sqrt{t}$; see Fig. 28.

This reduces the total energy to $E(\varphi_t^* \xi) \approx \frac{1}{t}$. However, some orbits in the shell stretch to the "full length" $\approx t$. Hence, the supremum norm $\|\varphi_t^* \xi\|_{L^\infty} = \max \|\xi_t\| = \mathcal{O}(t)$.

Once a length scale ℓ is selected, the energy cannot be squeezed to $< \frac{1}{\ell}$ by using the smooth one-parameter family. To proceed further, one has to renew the original stretching of the subball into the snake. This produces the next collapse at a finer scale. The corresponding 2-complex froth $K = \mathcal{F}_t$ "blossoms and branches" [Fr2]. The topology of K remains trivial (the froth is contractible to the boundary ∂B), since the complement to K is homeomorphic to a ball. $\square$

§4. Asymptotic linking number

The classical Hopf invariant for $S^3 \to S^2$-mappings has two definitions: a topological one (as the linking number of the preimages of two arbitrary points of S^2), and an integral one (as the value of $\int \omega \wedge d^{-1}\omega$ for any two-form ω on S^3 that is a pullback of a normalized area form on S^2); see Example 1.19.

The helicity of an arbitrary divergence-free vector field on a three-dimensional simply connected manifold is a straightforward generalization of the integral definition of the Hopf invariant. The topological counterpart is more subtle and leads to the notions of asymptotic and average linking numbers of field trajectories [Arn9], which replace the linking of the closed curves of the classical definition.

This section deals with such an ergodic interpretation of helicity.

4.A. Asymptotic linking number of a pair of trajectories

Let M be a three-dimensional closed simply connected manifold with volume element μ. Let ξ be a divergence-free field on M and $\{g^t : M \to M\}$ its phase flow.

Consider a pair of points x_1, x_2 in M. We will associate to this pair of points a number that characterizes the "asymptotic linking" of the trajectories of the flow $\{g^t\}$ issuing from these points. For this purpose, we first connect any two points x and y of M by a "short path" $\Delta(x, y)$. The conditions imposed on a system of short paths will be described below and are satisfied for "almost any" choice of the system.

We select two large numbers T_1 and T_2, and close the segments $g^t x_1$ ($0 \leq t \leq T_1$) and $g^t x_2$ ($0 \leq t \leq T_2$) of the trajectories issuing from x_1 and x_2 by the short paths $\Delta(g^{T_k} x_k, x_k)$ ($k = 1, 2$). We obtain two closed curves, $\Gamma_1 = \Gamma_{T_1}(x_1)$ and $\Gamma_2 = \Gamma_{T_2}(x_2)$; see Fig. 31. Assume that these curves do not intersect (which is true for almost all pairs x_1, x_2 and for almost all T_1, T_2). Then the linking number $lk_\xi(x_1, x_2; T_1, T_2) := lk(\Gamma_1, \Gamma_2)$ of the curves Γ_1 and Γ_2 is well-defined.
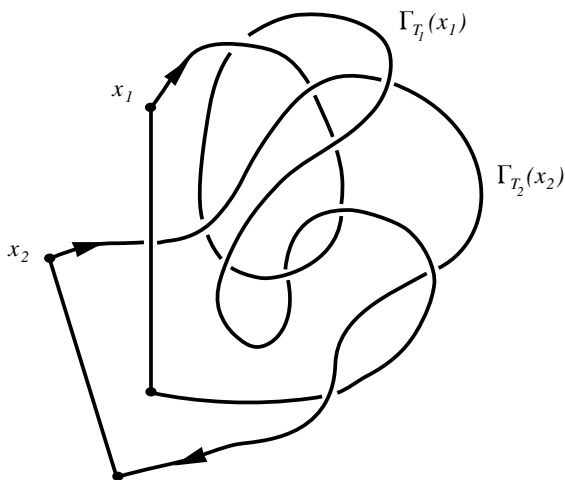
FIGURE 31.   The long segments of the trajectories are closed by the "short" paths.

Definition 4.1. The *asymptotic linking number* of the pair of trajectories $g^t x_1$ and $g^t x_2$ $(x_1, x_2 \in M)$ of the field ξ is defined as the limit

$$\lambda_\xi(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{lk_\xi(x_1, x_2; T_1, T_2)}{T_1 \cdot T_2},$$

where T_1 and T_2 are to vary so that Γ_1 and Γ_2 do not intersect.

Below we will see that this limit exists almost everywhere and is independent of the system of "short" paths Δ (as an element of the space $L_1(M \times M)$ of the Lebesgue-integrable functions on $M \times M$).

Definition 4.2. The *average (self-) linking number* of a field ξ is the integral over $M \times M$ of the asymptotic linking number $\lambda_\xi(x_1, x_2)$ of the field trajectories:

$$(4.1) \qquad \lambda_\xi = \int\limits_M \int\limits_M \lambda_\xi(x_1, x_2) \mu_1 \mu_2.$$

Remark 4.3. The average self-linking number can be defined via an auxiliary step by specifying what the asymptotic linking of field lines with a closed curve is and then by replacing the curve with another orbit. This approach is used in Section 5 to define the average crossing number.

Theorem 4.4 (Helicity Theorem, [Arn9]**).** *The average self-linking of a divergence-free vector field ξ on a simply connected manifold M with a volume element μ coincides with the field's helicity:*

$$(4.2) \qquad \lambda_\xi = \mathcal{H}(\xi).$$

Example 4.5. For the Hopf vector field $\mathbf{v}(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$ on the unit sphere $S^3 \subset \mathbb{R}^3$, the linking number of every two orbits (great circles) is equal to 1. All the orbits are periodic with the same period 2π. Hence, the value of $\lambda_\mathbf{v}(x_1, x_2)$, being the asymptotic linking of two trajectories per time unit, is $1/(4\pi^2)$. The average self-linking number of the Hopf field is

$$\lambda_\mathbf{v} = \int\limits_{S^3} \int\limits_{S^3} \lambda_\mathbf{v}(x_1, x_2) \mu_1 \mu_2 = \int\limits_{S^3} \int\limits_{S^3} \frac{1}{4\pi^2} \mu_1 \mu_2 = \frac{(\text{vol}(S^3))^2}{4\pi^2} = \frac{(2\pi^2)^2}{4\pi^2} = \pi^2,$$

which coincides with the mean helicity $\mathcal{H}(\mathbf{v})$ of the field $\mathbf{v}$; see Example 1.19.

Remark 4.6. The result can be literally generalized to the case of two different divergence-free fields ξ and η on a simply connected M. The linking number $\lambda_{\xi, \eta}(x, y)$ in the latter case measures the asymptotic linkage of the trajectories $g_\xi^t x$ and $g_\eta^t y$ issuing from x and y respectively. The helicity is replaced by the

bilinear form $\mathcal{H}(\xi, \eta)$; see Remark 1.22. The Helicity Theorem states in this case that

$$\int\limits_M \int\limits_M \lambda_{\xi,\eta}(x, y)\mu_x\mu_y = \int\limits_M \omega_\xi \wedge (d^{-1}\omega_\eta),$$

where the 2-forms are defined by $\omega_\xi = i_\xi\mu$, $\omega_\eta = i_\eta\mu$, and $d^{-1}\omega_\eta$ denotes an arbitrary potential 1-form α such that $d\alpha = \omega_\eta$.

In the case of a manifold M with boundary, all the vector fields involved are supposed to be tangent to the boundary.

Remark 4.7. The identity of the two classical definitions of the Hopf invariant (being a nonergodic version of the Helicity Theorem; see Example 1.19) is a manifestation of Poincaré duality.

Assume that we deal with singular forms (of δ-type) supported on compact submanifolds. Replace the differential forms by their supports. Then the operations d^{-1} and $\wedge$ correspond to the passage from the support submanifolds to the film bounded by them and to their intersections, respectively. Finally, the integration $\int_M$ is summation of the intersection points with the corresponding signs. The intersection of a submanifold with a film bounded by another submanifold is the linking number of these two submanifolds.

The consideration of smooth differential forms instead of singular ones leads to the averaging of appropriate linking characteristics. The asymptotic version of the linking number can be regarded in the context of *asymptotic cycles* [SchS, DeR, GPS, Sul].

A counterpart of the homotopy invariance of the classical Hopf invariant is unknown for the asymptotic linking number:

Problem 4.8. Is the average self-linking number of a divergence-free vector field invariant under the action of homeomorphisms preserving the measure on the manifold? Here, a measure-preserving homeomorphism is supposed to transform the flow of one smooth divergence-free vector field into the flow of the other, both fields having well-defined average self-linking numbers.

A partial (affirmative) answer to this question was given in [G-G], where the average linking number for a field in a solitorus was related to the topological invariants of Ruelle [Rue] and Calabi [Ca] for disk diffeomorphisms (see also Sections III.7.A and IV.8.B).

We will give two versions of the proof of the Helicity Theorem. The first one makes explicit use of the Gauss linking formula and of the Biot–Savart integral in $\mathbb{R}^3$. The second, coordinate-free, version reveals the reason for the helicity–linking coincidence on an arbitrary simply connected manifold.

Various generalizations of asymptotic linking are discussed in subsequent sections.

4.B. Digression on the Gauss formula

To state the formula given by Gauss for the linking number of two closed curves in three-dimensional Euclidean space, we introduce the following notation.

Let $\gamma_1 : S_1^1 \to \mathbb{R}^3$ and $\gamma_2 : S_2^1 \to \mathbb{R}^3$ be smooth mappings of two circumferences to $\mathbb{R}^3$ with disjoint images. Let $t_1 \pmod{T_1}$ and $t_2 \pmod{T_2}$ be coordinates on the first and second circumferences. We denote by $\dot{\gamma}_i = \dot{\gamma}_i(t_i)$, $i = 1, 2$, the corresponding velocity vectors in the images (Fig. 32).

Assume that the circumferences are oriented by the choice of the parameters t_1 and t_2, and fix an orientation for $\mathbb{R}^3$. Then we can define vector products and triple scalar products in $\mathbb{R}^3$.
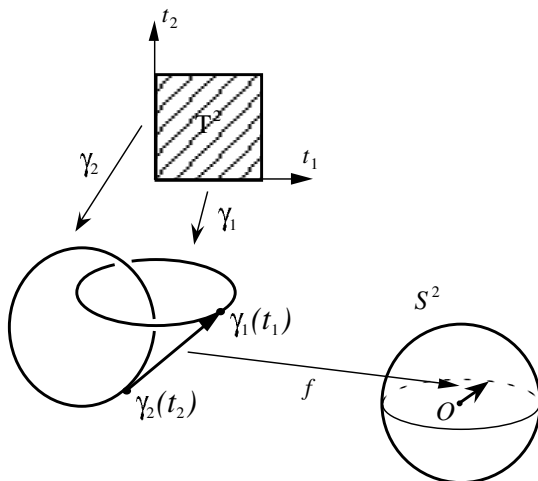
FIGURE 32.   Two parametrized linked curves in space define the Gauss map $T^2 \to S^2$.

Theorem 4.9 (Gauss Theorem). *The linking number of the closed curves $\gamma_1(S^1)$ and $\gamma_2(S^1)$ in $\mathbb{R}^3$ is equal to*

$$lk(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int\limits_0^{T_1} \int\limits_0^{T_2} \frac{(\dot{\gamma}_1, \dot{\gamma}_2, \gamma_1 - \gamma_2)}{\|\gamma_1 - \gamma_2\|^3} \, dt_1 dt_2.$$

Proof. Consider the mapping

$$f : \quad T^2 \to S^2$$

from the torus to the sphere sending a pair of points on our circumferences to the vector of unit length directed from $\gamma_2(t_2)$ to $\gamma_1(t_1)$: $f = F/\|F\|$, where $F(t_1, t_2) = \gamma_1(t_1) - \gamma_2(t_2)$; see Fig. 32.

We orient the sphere by the inner normal and the torus by the coordinates t_1, t_2.

Lemma 4.10. *The degree of the mapping f is equal to the linking number $lk(\gamma_1, \gamma_2)$.*

Indeed, this is true for small circumferences situated far away from each other: Both the linking number and the degree of the mapping f are 0; cf. Fig. 32. Neither of these quantities changes in the course of any deformation that leaves the curves disjoint. Furthermore, it is easy to verify that under any deformation of the pair of curves containing a passage of one curve through another, both the linking number and the degree of the mapping change by 1 with the same sign. Therefore, the equality $lk(\gamma_1, \gamma_2) = \deg f$ follows, in view of the connectedness of the set of smooth mappings $S^1 \to \mathbb{R}^3$.

Now the Gauss Theorem is a consequence of the following lemma.

Lemma 4.11. *The degree of the mapping $f \colon T^2 \to S^2$ is given by the Gauss integral formula.*

Proof of Lemma 4.11. By definition of the degree,

$$\deg f = \frac{1}{4\pi} \iint\limits_{T^2} f^* v^2,$$

where the 2-form v^2 is the area element on the unit sphere. Now, by definition of f, the value of the form $f^* v^2$ on a pair of vectors a_1, a_2 tangent to the torus at $t = (t_1, t_2) \in T^2$ is equal to their mixed product with the vector $-f := -f(t)$ (we oriented the sphere by means of the *inner* normal):

$$f^* v^2(a_1, a_2) = v^2(f_* a_1, f_* a_2) = (f_* a_1, f_* a_2, -f).$$

By differentiating f, we obtain $f_* a = F_* a / \|F\| + c(a, f)f$ (here $c(a, f)$ is a scalar factor), and therefore

$$v^2(f_* a_1, f_* a_2) = (F_* a_1, F_* a_2, -F) / \|F\|^3.$$

Recalling that $F = x_1 - x_2$, we obtain the expression

$$f^* \omega^2 = (\dot{x}_1, \dot{x}_2, x_1 - x_2) \|x_1 - x_2\|^{-3} dt_1 \wedge dt_2$$

for an element of the spherical image of the torus, as was to be shown.   □

The higher-dimensional version of the Gauss linking formula, developed in [Poh, Wh], is based on the same observation about equivalence of the linking and the degree of the Gauss map.

4.C. Another definition of the asymptotic linking number

Let $\{g^t\}$ be the phase flow defined by a divergence-free field ξ in a three-dimensional compact Euclidean domain $M \subset \mathbb{R}^3$. The field is assumed to be tangent to the boundary ∂M.

Define the Gauss linking of the ξ-trajectories as

$$\Lambda_\xi(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{1}{4\pi \cdot T_1 T_2} \int_0^{T_2} \int_0^{T_1} \frac{(\dot{x}_1(t_1), \dot{x}_2(t_2), x_1(t_1) - x_2(t_2))}{\|x_1(t_1) - x_2(t_2)\|^3} dt_1 dt_2,$$

where $x_i(t_i) = g^{t_i}(x_i)$ is the trajectory of the point x_i, and $\dot{x}_i(t_i) = \frac{d}{dt_i} g^{t_i} x_i$ is the corresponding velocity vector.

Lemma 4.12.

(1) *The limit $\Lambda_\xi(x_1, x_2)$ exists almost everywhere on $M \times M$.*
(2) *The value $\Lambda_\xi(x_1, x_2)$ coincides with the number $\lambda_\xi(x_1, x_2)$ defined above for almost all x_1, x_2.*

Proof. To prove the first statement, it is enough to verify that Λ is the "time average" of an integrable function on the manifold $M \times M$, on which the abelian group $\{g^{t_1}\} \times \{g^{t_2}\}$ acts. The integrand is the function

$$G(x_1, x_2) = \frac{(a_1, a_2, x_1 - x_2)}{\|x_1 - x_2\|^3},$$

where $a_k = \frac{d}{dt_k}|_{t_k=0} g^{t_k} x_k = \xi(x_k)$. The function G has a singularity on the diagonal of $M \times M$: It grows at most like r^{-2}, where r is the distance to the diagonal. Since the codimension of the diagonal is 3, the function G belongs to the space $L^1(M \times M)$, as required.

To compare Λ_ξ with λ_ξ, we represent the linking coefficient of the curves $\Gamma_1 = \Gamma_{T_1}(x_1)$ and $\Gamma_2 = \Gamma_{T_2}(x_2)$ by the Gauss integral with $0 \leq t_1 \leq T_1 + 1$, $0 \leq t_2 \leq T_2 + 1$, by using the value of the parameter t_k from T_k to $T_k + 1$ for parametrizing the "short path" $\Delta(g^{T_k} x_k, x_k)$ that joins $g^{T_k} x_k$ to x_k.

Definition 4.13. A *system of short paths* joining every two points in M is a system of paths depending in a measurable way on the points x and y in M and obeying the following condition. The integrals of Gauss type for every pair of nonintersecting paths of the system, and also for every nonintersecting pair (a path of the system, a segment of the phase curve $g^t x$, $0 \leq t \leq 1$), are bounded independently of the pair by a constant C.

Remark 4.14. One can verify that systems of short paths exist for nowhere vanishing vector fields or even for generic vector fields (with isolated zeros). It is useful to keep in mind that an integral of Gauss type for a pair of straight-line segments remains bounded when these segments approach each other. The phenomenon one has to avoid is the winding of a trajectory around a path of the system, which implies unboundedness of the integral. However, a small perturbation of the short path system leads to a system satisfying the condition above.

Indeed, the phenomenon of winding does not occur in systems where there is $N \in \mathbb{Z}_+$ such that at any point of the manifold M at least one of the derivatives of

Δ (along the paths) of order less than N does not coincide with that of g^t (along the flow). Given a vector field ξ (or equivalently, given flow g^t), the systems of short paths Δ subject to the latter constraint form an ample set (cf. the strong transversality theorem [AVG]).

The fields with nonisolated zeros constitute a set of infinite codimension in the space of all vector fields. For such vector fields, the existence question of systems of short paths is more subtle, and there still are some unresolved issues related to it.[1] It would be very interesting to complete the proof of existence in full generality.

Now, the difference $\int_0^{T_2+1}\int_0^{T_1+1} - \int_0^{T_2}\int_0^{T_1}$ of Gauss-type integrals can be estimated by the sum of at most $[T_1] + [T_2] + 1$ terms, none of which exceeds C. Therefore,

$$\lambda_\xi(x_1, x_2) - \Lambda_\xi(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{1}{4\pi \cdot T_1 T_2} \left(\int_0^{T_2+1}\int_0^{T_1+1} - \int_0^{T_2}\int_0^{T_1} \right) = 0$$

(where T_1 and T_2 tend to infinity over any sequence for which the curves $\Gamma_1 = \Gamma_{T_1}(x_1)$ and $\Gamma_2 = \Gamma_{T_2}(x_2)$ do not meet). $\qquad\square$

Now we complete the proof of the Helicity Theorem on the equivalence of the ergodic and integral definitions of the helicity of a divergence-free vector field defined in a domain $M \subset \mathbb{R}^3$ (and tangent to the boundary ∂M).

Consider the Biot–Savart integral

$$A(x_2) = -\frac{1}{4\pi} \int_M \frac{\xi(x_1) \times (x_1 - x_2)}{\|x_1 - x_2\|^3} \mu(x_1)$$

(where $\times$ denotes the cross product) that defines a vector-potential $A = \mathrm{curl}^{-1} \xi$ in $\mathbb{R}^3$. It allows one to obtain the integral representation of the helicity

$$\mathcal{H}(\xi) = \langle \xi, \mathrm{curl}^{-1} \xi \rangle = \langle \xi, A \rangle = \frac{1}{4\pi} \iint_{M \times M} \frac{(\xi(x_1), \xi(x_2), x_1 - x_2)}{\|x_1 - x_2\|^3} \mu(x_1)\mu(x_2).$$

The Helicity Theorem follows from this formula and from the Birkhoff ergodic theorem applied to the integrable function $(\xi(x_1), \xi(x_2), x_1 - x_2)/(4\pi\|x_1 - x_2\|^3)$ on $M \times M$. The space average

$$\frac{\iint\limits_{M \times M} \Lambda_\xi(x_1, x_2)\mu(x_1)\mu(x_2)}{(\mathrm{vol}(M))^2} = \frac{\lambda_\xi}{(\mathrm{vol}(M))^2}$$

of the time average Λ_ξ along the trajectories of the measure-preserving flow of ξ coincides with the space average $\mathcal{H}(\xi)/(\mathrm{vol}(M))^2$ of the function. $\qquad\square$

---

[1] We are grateful to P. Laurence, who noted that an existence proof would require some kind of "global approach," considering vector fields on the whole manifold, while the transversality theorem is "local" in nature.

Remark 4.15. Note that for an ergodic field (ξ, ξ) on $M \times M$ the function $\Lambda_\xi(x_1, x_2)$ is constant almost everywhere: The asymptotic linking numbers for almost all pairs of ξ-trajectories are equal to each other.

4.D. Linking forms on manifolds

Here we show how the preceding arguments can be adjusted to the case of an arbitrary simply connected manifold, where the Gauss-type integral of De Rham's "double form" [DeR] cannot be written as explicitly as in $\mathbb{R}^3$ (see [KhC]).

Theorem 4.16 (= 4.6′). *The average linking number of two divergence-free vector fields ξ and η coincides with $\mathcal{H}(\xi, \eta)$:*

$$\iint\limits_{M \times M} \lambda_{\xi,\eta}(x, y)\mu_x\mu_y = \int_M i_\xi\mu \wedge d^{-1}(i_\eta\mu).$$

Proof. We start by recalling some facts about double bundles and linking forms. Denote by $\Omega^k(M)$ the space of differential k-forms on a manifold M.

Definition 4.17. A differential 2-form $G \in \Omega^2(M \times M)$ is called a *Gauss–De Rham linking form* on a simply connected manifold M if for an arbitrary pair of nonintersecting closed curves Γ_1 and Γ_2 the integral of this form over $\Gamma_1 \times \Gamma_2$ equals the corresponding linking number:

$$\iint\limits_{\Gamma_1 \times \Gamma_2 \subset M \times M} G = lk(\Gamma_1, \Gamma_2).$$

Here $\Gamma_1 \times \Gamma_2 = \{(x, y) \in M \times M \mid x \in \Gamma_1, y \in \Gamma_2\}$. The existence of such a form will be established later.

Definition 4.18. Each differential form $K(x, y) \in \Omega^*(M \times M)$ determines an *operator* $\tilde{K} : \Omega^*(M) \to \Omega^*(M)$ on the space of differential forms $\Omega^*(M)$ on M that sends a differential form $\varphi(y)$ into the differential form

$$(\tilde{K}\varphi)(x) = \int\limits_{\pi^{-1}(x)} K(x, y) \wedge \varphi(y),$$

where $\pi : M \times M \to M$ is the projection on the first component, and the integration is carried out over the fibers of this projection; see Fig. 33. The value of the form $\tilde{K}\varphi$ at a point $x \in M$ is the integral over the fiber $\pi^{-1}(x) \subset M \times M$ of the wedge product $K(x, y) \wedge \varphi(y)$. If the product $K(x, y) \wedge \varphi(y)$ is an n-form in y, then by definition, $(\tilde{K}\varphi)(x) = 0$.

Proposition 4.19. *The operator $\widetilde{G}$ corresponding to the linking form is the Green operator inverse to the exterior derivative of 1-forms: If $\psi = d\varphi$ and $\varphi \in \Omega^1(M)$,*
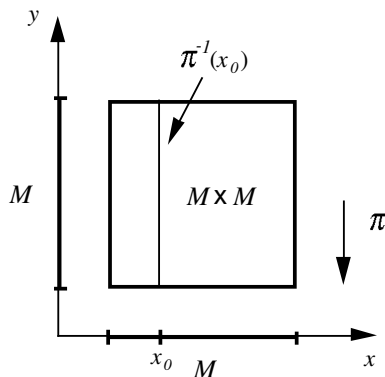
FIGURE 33.   Any form on $M \times M$ defines an operator on $\Omega^*(M)$.

then

$$\varphi = \widetilde{G}(\psi) + dh$$

for a certain function h.

The term dh materializes the fact that a potential 1-form φ can be reconstructed from an exact 2-form ψ modulo a full differential only.

Proof of Proposition. Let $\mathbf{d} = d_x + d_y$ be the operator of the exterior derivative on $\Omega^*(M \times M)$.

Lemma 4.20. $\widetilde{d_x K} = d \circ \widetilde{K}$.

Indeed, $[d_x K(x, y)] \wedge \varphi(y) = d_x[K(x, y) \wedge \varphi(y)]$, and hence

$$\int_{\pi^{-1}(x)} [d_x K(x, y)] \wedge \varphi(y) = d \left(\int_{\pi^{-1}(x)} K(x, y) \wedge \varphi(y) \right).$$

Lemma 4.21. *If K is a 1-form in the variable y, then* $\widetilde{d_y K} = \widetilde{K} \circ d$.

This follows from the identity

$$\int_{\pi^{-1}(x)} [d_y K(x, y)] \wedge \varphi(y) = \int_{\pi^{-1}(x)} K(x, y) \wedge d\varphi(y).$$

Lemma 4.22. *The exterior derivative of a Gauss–De Rham form G on $M \times M$ is the sum $\mathbf{d}G = \delta + \beta$ of*

- *the δ-form on the diagonal* $\Delta \subset M \times M$ *(the integral of the δ-form over any 3-chain in* $M \times M$ *is equal to the algebraic number of intersection points of the chain with the diagonal* Δ*), and of*
- *some form* $\beta \in \Omega^3(M \times M)$ *that is a linear combination of forms from* $\Omega^k(M) \otimes \Omega^{3-k}(M)$*-forms with each factor being closed.*

Proof.

$$lk(\Gamma_1, \Gamma_2) = \iint\limits_{\Gamma_1 \times \Gamma_2} G = \iiint\limits_{\partial^{-1}(\Gamma_1 \times \Gamma_2)} \mathbf{d}G = \iiint\limits_{(\partial^{-1}\Gamma_1) \times \Gamma_2} \mathbf{d}G.$$

On the other hand, since the linking number is the intersection number of the cycle Γ_2 and a surface $\partial^{-1}\Gamma_1$ (whose boundary is Γ_1), it can be represented as the integral of the δ-form over the chain $(\partial^{-1}\Gamma_1) \times \Gamma_2$:

$$lk(\Gamma_1, \Gamma_2) = \iiint\limits_{(\partial^{-1}\Gamma_1) \times \Gamma_2} \delta.$$

Now the statement follows from the fact that all those β's are closed, and each β is characterized by the conditions

$$\iiint\limits_{(\partial^{-1}\Gamma_1) \times \Gamma_2} \beta = 0 \qquad \text{and} \qquad \iiint\limits_{\Gamma_1 \times (\partial^{-1}\Gamma_2)} \beta = 0.$$

$\square$

Remark 4.23. The form β can be chosen in such a way that the cohomology class of $\delta + \beta$ in $H^3(M \times M)$ is trivial. Indeed, though the class of δ in $H^3(M \times M) = \sum_k H^k(M) \otimes H^{3-k}(M)$ is nontrivial (the diagonal in $M \times M$ is not a boundary), adding an appropriate β we can get rid of the $H^0(M)$ and $H^3(M)$ terms. Hence, the class of $\delta + \beta$ vanishes due to the simple-connectedness of M ($H^1(M) = H^2(M) = 0$).

This proves the existence of a Gauss–De Rham linking 2-form G as a solution of the equation $[\mathbf{d}G] = 0 \in H^3(M \times M)$, where $[*]$ denotes the cohomology class of a differential form.

To complete the proof of Proposition 4.19, we pass from the equation on forms $\mathbf{d}G = \delta + \beta$ to the relation on the corresponding operators: $\widetilde{\mathbf{d}G} = \tilde{\delta} + \tilde{\beta}$, or

$$\widetilde{d_x G} + \widetilde{d_y G} = \tilde{\delta} + \tilde{\beta}.$$

At this point we notice that

(a) the δ-form corresponds to the identity operator $\tilde{\delta} = \text{Id}$, and
(b) the image of the operator $\tilde{\beta}$ in $\Omega^*(M)$ belongs to the subspace of closed forms (see Lemma 4.22). In particular, within $\Omega^1(M)$ the image consists of the exact forms.

Combining these facts with Lemmas 4.20–21, we come to the relation

$$d \circ \widetilde{G} + \widetilde{G} \circ d = \mathrm{Id} + d \circ \tilde{\gamma}$$

for operators on one-forms in M. Having applied the operators of both sides of the relation to a form φ and rearranging the terms, one transforms this relation into

$$d \circ (\widetilde{G}(\varphi) - \tilde{\gamma}(\varphi)) + \widetilde{G}(d\varphi) = \varphi.$$

Finally, for $\psi = d\varphi$, we obtain

$$\varphi = \widetilde{G}(\psi) + dh$$

for some function h.   □

Lemma 4.24. *There exists a Gauss–De Rham linking form $G(x, y)$ with a pole of order 2 on the diagonal of $M \times M$.*

Proof. The linking number of Γ_1 with Γ_2 by definition coincides with the linking number of $\Gamma_1 \times \Gamma_2$ with the diagonal Δ in $M \times M$. Identify a neighborhood of the diagonal in $M \times M$ with a neighborhood of the zero section in the normal bundle $T^\perp \Delta$ over the diagonal via the geodesic exponential map (Fig. 34).
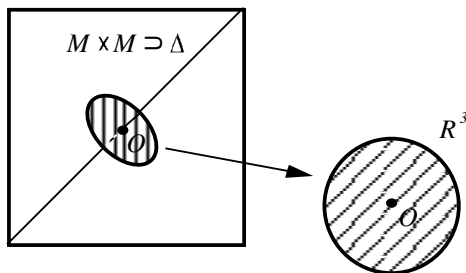
FIGURE 34. For any point of the diagonal $\Delta \subset M \times M$ a neighborhood in the transversal to Δ direction can be identified with a neighborhood in $\mathbb{R}^3$.

Then, in every fiber (being a neighborhood of $0 \in \mathbb{R}^3$), we consider the standard Gauss linking form singular at the origin. The latter is the 2-form obtained by the substitution of the radius vector field $\nabla(1/r)$ into the standard volume element in $\mathbb{R}^3$. It has a pole of order 2 at the origin. Extend the definition of this form from one fiber to the entire neighborhood of Δ in $M \times M$ by prescribing that this form vanishes on vectors parallel to $\Delta \subset T^\perp \Delta$. We obtain a linking form in $M \times M$ that has a pole of the desired order 2 on the diagonal.   □

Corollary 4.25. *The linking form G is integrable: $G \in L^1(M \times M)$; i.e., the value of G evaluated on any two smooth vector fields is an integrable function on $M \times M$.*

Indeed, the codimension of the diagonal in $M \times M$ equals 3, and the growth order of the form G near the diagonal is 2.

Remark 4.26. All the above arguments on the Gauss–De Rham linking forms hold (with certain evident modifications) for manifolds of arbitrary dimension. Further consideration in this section is essentially three-dimensional.

Let ξ and η be divergence-free fields on M equipped with a volume form μ. Let $g_\xi^t x$ and $g_\eta^s y$ be the segments of the trajectories of these fields starting at x and y for time intervals $0 \le t \le T$ and $0 \le s \le S$. Denote by Δ_x and Δ_y the corresponding "short paths" closing the segments of the trajectories and making them into two piecewise smooth closed curves.

The asymptotic linking number is equal to

$$\lambda_{\xi,\eta}(x,y) = \lim_{T,S \to \infty} \frac{1}{T \cdot S} \iint_{(g_\xi^t x \cup \Delta_x) \times (g_\eta^s y \cup \Delta_y)} G = \lim_{T,S \to \infty} \frac{1}{T \cdot S} \iint_{g_\xi^t x \times g_\eta^t y} G.$$

The last equality of the limits follows from the boundedness of the integrals over the short paths (see Definition 4.13 of a short paths system). Hence,

$$\lambda_{\xi,\eta} = \iint_{M \times M} \lambda_{\xi,\eta}(x,y) \mu_x \mu_y = \int_M \mu_x \int_M \mu_y \left(\lim_{T,S \to \infty} \frac{1}{T \cdot S} \iint_{g_\xi^t x \times g_\eta^t y} G \right)$$

$$= \int_M \mu_x \int_M \mu_y \left(\lim_{T,S \to \infty} \frac{1}{T \cdot S} \int_0^T \int_0^S (i_\xi i_\eta G)\, ds\, dt \right),$$

where $i_\xi i_\eta G$ is regarded as a function on $M \times M$ and $\int_0^T \int_0^S$ denotes the integral of this function over the product of (the pieces of) the trajectories $g_\xi^t x$ and $g_\eta^s y$.

By the Birkhoff ergodic theorem applied to the integrable function $i_\xi i_\eta G$, we can pass from the time averages to the space average:

$$\lambda_{\xi,\eta} = \iint_{M \times M} \lambda_{\xi,\eta}(x,y) \mu_x \mu_y = \int_M \mu_x \int_M \mu_y (i_\xi i_\eta G).$$

Finally, shift the substitution operators i_ξ and i_η from G to the forms μ_x and μ_y (the operation i_ξ is inner differentiation; see Section 1):

$$\lambda_{\xi,\eta} = \iint_{M \times M} \lambda_{\xi,\eta}(x,y)\, \mu_x \mu_y = \int_M \mu_x \int_M \mu_y (i_\xi i_\eta G) = \int_M i_\xi \mu_x \wedge \left(\int_M i_\eta \mu_y \wedge G \right)$$

$$= \int_M i_\xi \mu \wedge \widetilde{G}(i_\eta \mu).$$

By Proposition 4.19 the operator $\widetilde{G}$ is inverse to exterior differentiation: $\widetilde{G}(i_\eta\mu) = d^{-1}(i_\eta\mu)$ modulo an exact form. This completes the proof of Theorem 4.16:

$$\lambda_{\xi,\eta} = \int_M i_\xi\mu \wedge d^{-1}(i_\eta\mu) = \mathcal{H}(\xi, \eta).$$

$\square$

§5. Asymptotic crossing number

The helicity approach to magnetic energy minoration in terms of the topology of magnetic lines was generalized by Freedman and He [FH1, 2] by introducing the notion of asymptotic crossing number. They determined the complexity of a knotted orbit by the "minimal number of crossings" in its projections. It replaces the linking number, where the crossings are counted with appropriate signs. In the presentation below we mostly follow the paper [FH2].

5.A. Energy minoration for generic vector fields

Definition 5.1. For two closed curves γ_1 and γ_2 in $\mathbb{R}^3$ the *crossing number* $c(\gamma_1, \gamma_2)$ is equal to the integral of the absolute value of the Gauss integrand for their linking number:

$$(5.1) \qquad c(\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_0^{T_1} \int_0^{T_2} \frac{|(\dot{\gamma}_1, \dot{\gamma}_2, \gamma_1 - \gamma_2)|}{\|\gamma_1 - \gamma_2\|^3} dt_1 dt_2.$$

This quantity is no longer invariant under a curve isotopy. However, all the notions and definitions regarding the corresponding asymptotic version can be literally transferred to this situation.

For a vector field ξ defined in a domain $M \subset \mathbb{R}^3$ (and tangent to the boundary ∂M), we use the same definition of a "system of short paths" as above (see Definition 4.13 and subsequent Remark 4.14). Denote by $\Gamma_T(x)$ the piece of the ξ-orbit of $x \in M$ run in the time period $[0, T]$ and closed by a short path.

Definition 5.2. The *asymptotic crossing number* of the field lines of a divergence-free vector field ξ with a closed curve γ in a simply connected manifold M^3 is the limit

$$c_\xi(x, \gamma) = \lim_{T\to\infty} \sup \frac{1}{T} c(\Gamma_T(x), \gamma).$$

This limit exists, belongs to $L^1(M)$, and is well-defined in $L^1(M)$ in spite of the ambiguity in the choice of the system of short curves.

Similarly, the *average crossing number* of the field lines of ξ with the curve γ is given by the integral

$$c_\xi(\gamma) = \int_M c_\xi(x, \gamma)\mu_x,$$

where μ is a volume form on M.

Finally, given two divergence-free vector fields ξ and η, their *asymptotic crossing number $Cr(\xi, \eta)$* is defined as the space integral of the crossing number of one of the fields with the trajectories of the other:

$$Cr(\xi, \eta) = \int_M (\limsup_{T \to \infty} \frac{1}{T} c_\xi(\Gamma_T(y))\mu_y,$$

where $\Gamma_T(y)$ is the piece $0 \le t \le T$ of the ξ-field line issuing from the point y and closed by a short path. This crossing number admits the integral representation

$$(5.2) \qquad Cr(\xi, \eta) = \frac{1}{4\pi} \iint_{M \times M} \frac{|(\xi(x), \eta(y), x - y)|}{\|x - y\|^3} \mu_x \mu_y.$$

The asymptotic crossing number yields the following lower bound for the $E_{3/2}$-*energy* $E_{3/2}(\xi) := \int_M \|\xi\|^{3/2}\mu$.

Theorem 5.3 [FH2]. *For any divergence-free vector field ξ in M*

$$(5.3) \qquad E_{3/2}(\xi) \ge \left(\frac{16}{\pi}\right)^{1/4} Cr(\xi, \xi)^{3/4}.$$

Remarks 5.4 [FH2]. **(A)** The $L^{3/2}$-norm used in the definition of the $E_{3/2}$-energy is justified by the "conformal nature" of the problem. Any lower bound for the $E_{3/2}$-energy implies a lower bound for the standard E_2-energy $E_2(\xi) := \int_M \|\xi\|^2\mu$ due to a straightforward application of the Hölder inequality:

$$(5.4) \qquad E_2(\xi) \ge \frac{(E_{3/2}(\xi))^{4/3}}{(\mathrm{vol}(M))^{1/3}} \ge \left(\frac{16}{\pi \cdot \mathrm{vol}(M)}\right)^{1/3} Cr(\xi, \xi),$$

or, in a more recognizable form, $\int(\|\xi\|^{3/2} \cdot 1) \le (\int \|\xi\|^2)^{3/4} \cdot (\mathrm{vol}(M))^{1/4}$.

(B) Similarly, for any two divergence-free vector fields ξ and η in M,

$$Cr(\xi, \eta) \le \left(\frac{16}{\pi}\right)^{1/4} \left(E_{3/2}(\xi)\right)^{2/3} \cdot \left(E_{3/2}(\eta)\right)^{2/3}.$$

(C) Both sides of the inequality have geometric nature (they rely on a particular choice of metric) and are not topologically invariant. On the other hand, the energy estimate in terms of the helicity gives a topological bound for a geometric quantity.

One can make the right-hand side of the inequality (5.3) topological by brute force, defining the *topological crossing number*

$$Cr_{\text{top}}(\xi, \eta) = \inf_{h \in \text{Diff}(\mathbb{R}^3)} Cr(h_*\xi, h_*\eta).$$

Then

$$E_{3/2}(h_*\xi) \geq \left(\frac{\pi}{16}\right)^{1/4} Cr_{\text{top}}(\xi, \xi)^{3/4},$$

for any $h \in \text{Diff}(\mathbb{R}^3)$.

(D) Theorem 5.3 holds for vector fields with an arbitrary divergence, provided that $Cr(\xi, \xi)$ is defined by the integral formula (5.2) and not ergodically. Having used the integral definition of the helicity as well (see Definition 1.3), one obtains

$$E_{3/2}(h_*\xi) \geq \left(\frac{\pi}{16}\right)^{1/4} |\mathcal{H}(\xi)|^{3/4}, \qquad \text{for any} \quad h \in \text{Diff}(M \subset \mathbb{R}^3),$$

by virtue of the evident inequality $Cr(\xi, \xi) \geq |\mathcal{H}(\xi)|$.

Remark 5.5. A two-dimensional version of the asymptotic crossing number has been developed and applied to energy estimates of the braided magnetic tubes in [Be2]. In this case the energy lower bound appears to be quadratic in the total crossing number of a braided field, while the energy of a knotted field in three-dimensional space is bounded by an expression linear in Cr (see the estimate (5.4) for the E_2-energy above).

Proof of Theorem 5.3. The integral form of the asymptotic crossing number yields the following upper bound:

$$Cr(\xi, \xi) = \frac{1}{4\pi} \int_M \int_M \frac{|(\xi(x), \xi(y), x - y)|}{\|x - y\|^3} \mu_x \mu_y$$

$$\leq \frac{1}{4\pi} \int_M \int_M \|\xi(y)\| \left(\frac{\|\xi(x)\|}{\|x - y\|^2}\right) \mu_x \mu_y = \int_M \|\xi(y)\| \, \rho(y)\mu_y,$$

where the density $\rho : \mathbb{R}^3 \to \mathbb{R}^+$ is defined as

$$\rho(y) = \frac{1}{4\pi} \int_M \frac{\|\xi(x)\|}{\|x - y\|^2} \mu_x.$$

By the Hardy–Littlewood–Sobolev inequality [Sob1, Lieb] in potential theory, one obtains

$$\|\rho\|_{L^3} = \left(\int_M \rho^3 \mu\right)^{1/3} \leq \left(\frac{\pi}{16}\right)^{1/3} \left(\int_M \|\xi\|^{3/2}\mu\right)^{2/3}.$$

After combining it with Hölder's inequality one sees that

$$Cr(\xi, \xi) \leq \int_M \|\xi(y)\| \, \rho(y) \mu_y$$

$$\leq \|\xi\|_{L^{3/2}} \cdot \|\rho\|_{L^3} \leq \left(\frac{\pi}{16}\right)^{1/3} (\|\xi\|_{L^{3/2}})^2,$$

and the theorem follows. □

5.B. Asymptotic crossing number of knots and links

Apparently, any reasonably sharp estimates of Cr_{top} for a fairly generic field ξ are beyond reach. However, much more can be done under the (already exploited) assumption that the vector field has some linked or knotted invariant tori.

Definitions 5.6. The *crossing number $cn(K)$* (or $cn(L)$) of a knot K (or link L) in $\mathbb{R}^3$ is the minimum number of crossings of all plane diagrams representing the knot (or the link).

Consider some tubular neighborhood $\mathbf{T}$ of the (oriented) knot K. An arbitrary closed oriented curve confined to the neighborhood is said to be of *degree p* if it can be isotoped within $\mathbf{T}$ to the curve that is K covered p times.

A two-component link (P, Q) in $\mathbb{R}^3$ is called a *degree (p, q) satellite link* of K (p and q are positive integers) if (P, Q) can be (simultaneously) isotoped to a pair of curves $(P', Q') \subset \mathbf{T}$ with degree$(P') = p$ and degree$(Q') = q$. The *over-crossing number $cn(P, Q)$* of the link (P, Q) is defined to be the minimum number of overcrossings of P over Q among all planar knot diagrams representing (P, Q); see Fig. 35.

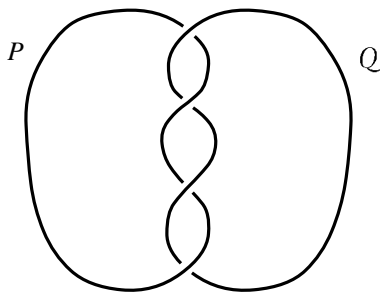FIGURE 35.   The crossing number of this link $L = P \cup Q$ is $cn(L) = 4$. The over-crossing number is $cn(P, Q) = 2$.

Let $cn_{p,q}(K)$ be the minimum of $cn(P, Q)$ over all degree (p, q) satellite links (P, Q) of K. Define the *asymptotic crossing number* of the knot K to be

$$(5.5) \qquad ac(K) = \lim_{p,q \to \infty} \inf \, cn_{p,q}(K)/pq = \inf\{cn_{p,q}(K)/pq \mid p, q \geq 1\}.$$

Remark 5.7. The equivalence of the two definitions of $ac(K)$ follows from the construction of an analogue of a k-fold alternate diagram for a degree (p, q) satellite that represents a (kp, kq) satellite. The number of crossings of the (smartly chosen) degree (kp, kq) satellite differs from that of the degree (p, q) satellite by the factor k^2; see [FH2].

Obviously, $ac(K) \leq cn(K)$, since $cn(P, Q) \leq cn(K)$ for P and Q taken to be copies of a minimal knot diagram.

Conjecture 5.8 [FH2]. $ac(K) = cn(K)$.

Theorem 5.9 [FH2]. *For a divergence-free field ξ defined in the solid torus $\mathbf{T}$ of knot type K and parallel to the boundary $\partial\mathbf{T}$ one has the inequality*

$$Cr(\xi, \xi) \geq |\operatorname{Flux}(\xi)|^2 ac(K).$$

Corollary 5.10. $Cr_{\text{top}}(\xi, \xi) \geq |\operatorname{Flux}(\xi)|^2 ac(K).$

Corollary 5.11. *The $E_{3/2}$-energy of such a field ξ yields the following lower bound:*

$$E_{3/2}(\xi) \geq \left(\frac{16}{\pi}\right)^{1/4} |\operatorname{Flux}(\xi)|^{3/2} (ac(K))^{3/4}.$$

Proof. Combine the above with Theorem 5.3. □

Notice that the right-hand side of the energy inequality is now topologically invariant.

The estimate can be specified even further in terms of knot invariants (we refer to [FH2] for the details and the proofs). A *Seifert surface* of a knot $K \in \mathbb{R}^3$ is an arbitrary surface embedded in $\mathbb{R}^3$ whose boundary is the knot K. The *genus* of a knot is the minimal genus (number of handles pasted to a disk) of an oriented Seifert surface. By the very definition, the genus is at least 1 for nontrivial knots (an unknot bounds a genuine embedded disk).

Theorem 5.12 [FH2]. *For any knot K the asymptotic crossing number $ac(K)$ satisfies $ac(K) \geq 2 \cdot genus(K) - 1$. In particular, $ac(K) \geq 1$ for a nontrivial knot.*

Definition 5.13. For a link $L = (L_1, \ldots, L_k)$, one first chooses a neighborhood consisting of k solitori $\mathbf{T}_1, \ldots, \mathbf{T}_k$ disjointly embedded in $\mathbb{R}^3$. Introduce quantities $cn_{p,q}(L_i; L)$, $i \in \{1, \ldots, k\}$ to be the minimal number of times a curve of degree p in $\mathbf{T}_i$ must pass over (when projected into a plane) a k component link created by choosing degree one curves in $\mathbf{T}_1, \ldots, \mathbf{T}_k$. Similarly, one defines the *asymptotic*

crossing number $ac(L_i, L)$ of L_i over L by formula (5.5), with the replacement of $cn_{p,q}(K)$ by $cn_{p,q}(L_i; L)$.

Then for a divergence-free field ξ leaving invariant the link of solid tori,

$$Cr_{\text{top}}(\xi, \xi) \geq \left(\sum_{i=1}^{k} ac(L_i, L) \cdot |\operatorname{Flux}(\xi|_{\mathbf{T}_i})| \right) \cdot \min_{1 \leq j \leq k} \{|\operatorname{Flux}(\xi|_{\mathbf{T}_j})|\}.$$

In particular, for a two-component link of solid tori $(\mathbf{T}_1, \mathbf{T}_2)$, one can deduce that

$$Cr_{\text{top}}(\xi, \xi) \geq 2|lk(L_1, L_2) \cdot \operatorname{Flux}(\xi|_{\mathbf{T}_1}) \cdot \operatorname{Flux}(\xi|_{\mathbf{T}_2})|.$$

Thus certain energy minorations can be obtained from the solution of a purely topological problem of the calculation of the quantities $ac(K)$ and $ac(L_i, L)$ for given types of knots and links of vortex tubes [FH2].

Remark 5.14. These invariants are finer than the linking numbers, due to the following immediate corollary of the plane projection method of computation of linking numbers:

$$ac(L_i, L) \geq \sum_{i \neq j} |lk(L_i, L_j)|, \quad 1 \leq i \leq k.$$

This estimate is useless for configurations with vanishing linking numbers (as the Borromean rings; Fig. 26b). A statement similar to Theorem 5.12 provides a lower bound for $ac(L_i, L)$ in terms of the so-called Thurston norm of certain surfaces associated to a link L (see [FH2]). In particular, if L_i is not a trivial component split away from the rest of the link L (say, L_i is one of components of the Borromean rings), then the asymptotic crossing number $ac(L_i, L)$ is minorized by 1.

Proof of Theorem 5.9. Define the *degree* of a (multivalued) function $f : \mathbf{T} \to S^1 = \mathbb{R}/\mathbb{Z}$ to be its homological degree, i.e., the winding number of the function on the solitorus.

Lemma 5.15. *For a vector field ξ parallel to the boundary $\partial \mathbf{T}$ of a solitorus $\mathbf{T}$*

$$(5.6) \qquad \operatorname{Flux}(\xi) = \int_{\mathbf{T}} (\xi, \nabla f) \mu,$$

for any degree 1 function $f : \mathbf{T} \to \mathbb{R}/\mathbb{Z}$.

Proof of Lemma. Cut the solid torus $\mathbf{T}$ along any surface Σ (representing the generator of $H_2(\mathbf{T}, \partial \mathbf{T})$; Fig. 36) to form a cylinder F.

The function f on $\mathbf{T}$ gives rise to a function $\tilde{f} : F \to \mathbb{R}$ on the cylinder. The values of $\tilde{f}$ at the corresponding points of the cylinder top $\partial_+ F$ and bottom $\partial_- F$
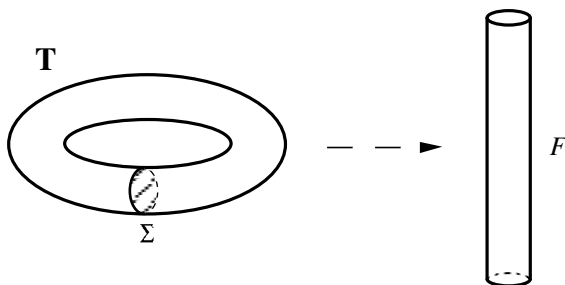
FIGURE 36.   Cut the solid torus along Σ to obtain a cylinder.

differ by 1. Denote by dA the area element on the section Σ. Then

$$
\begin{aligned}
\int_{\mathbf{T}} (\xi, \nabla f)\mu &= \int_{F} (\xi, \nabla \tilde{f})\mu = \int_{F} \operatorname{div}(\tilde{f}\xi)\mu \\
&= \int_{\partial_+ F} (\tilde{f}\xi, n)\, dA + \int_{\partial_- F} (\tilde{f}\xi, n)\, dA \\
&= \int_{\Sigma} ([\tilde{f}(\text{top}(x)) - \tilde{f}(\text{bottom}(x)]\xi, n)\, dA(x) = \int_{\Sigma} (\xi, n)\, dA.
\end{aligned}
$$

The lemma is proved. $\square$

To prove the theorem, we assume that $\operatorname{Flux}(\xi) = 1$, and g^t is the phase flow of ξ. Then, for a fixed C^1-mapping $f : \mathbf{T} \to \mathbb{R}/\mathbb{Z}$ of degree 1 and its lift $\tilde{f} : \tilde{\mathbf{T}} \to \mathbb{R}$,

$$
(5.7) \qquad \int_{\mathbf{T}} (\tilde{f}(g^\tau(x)) - \tilde{f}(x))\mu = \int_0^\tau \int_{\mathbf{T}} (\nabla f(g^t(x)), \xi(g^t(x)))\mu_x dt
$$

$$
= \int_0^\tau \operatorname{Flux}(\xi)\, dt = \tau.
$$

Recall that $\Gamma_\tau(x)$ is the curve $g^t(x), 0 \le t \le \tau$, joined to the "short curve" $\Delta(g^\tau(x), x)$ for any $x \in \mathbf{T}$. Then

$$
|\deg(\Gamma_\tau(x)) - (\tilde{f}(g^\tau(x)) - \tilde{f}(x))| \le C,
$$

since the lengths of the short paths are uniformly bounded and the function $\tilde{f}$ is continuously differentiable.

On the other hand, by definition of the asymptotic crossing number,

$$
(5.8) \qquad c(\Gamma_\tau(x), \gamma) \ge ac(K) \cdot \deg(\Gamma_\tau(x)) \cdot \deg(\gamma)
$$

for any closed curve γ in the solitorus $\mathbf{T}$. Therefore,

$$
c(\Gamma_\tau(x), \gamma) \ge ac(K) \cdot \deg(\gamma) \cdot [(\tilde{f}(g^\tau(x)) - \tilde{f}(x)) - C].
$$

Combining this inequality with formula (5.7), we obtain

$$\frac{1}{\tau} \int_{\mathbf{T}} c(\Gamma_\tau(x), \gamma)\mu_x \geq ac(K) \cdot \text{degree}(\gamma)\left(1 - \frac{C \cdot \text{vol}(\mathbf{T})}{\tau}\right).$$

Finally, as $\tau \to \infty$ it bounds below the average crossing number $c_\xi(\gamma)$:

$$c_\xi(\gamma) \geq ac(K) \cdot \text{degree}(\gamma).$$

Similarly, letting $\gamma = \Gamma_\tau(y)$, $y \in \mathbf{T}$, and utilizing formula (5.8) and the definition of the asymptotic crossing number, we deduce the required inequality

$$Cr(\xi, \xi) \geq ac(K).$$

<div style="text-align: right">□</div>

5.C. Conformal modulus of a torus

Some energy bounds for vector fields possessing invariant tori can be formulated in terms of the conformal modulus of a solid torus.

Let $\mathbf{T}$ be a solitorus endowed with some Riemannian metric. Homotopically $\mathbf{T}$ is equivalent to the circle $S^1 = \mathbb{R}/\mathbb{Z}$.

Definition 5.16. The *conformal modulus* of a solitorus $\mathbf{T}$ with a metric on it is

$$m(\mathbf{T}) = \inf_f \int_{\mathbf{T}} \|\nabla f\|^3 \mu,$$

where $f : \mathbf{T} \to \mathbb{R}/\mathbb{Z}$ is taken to be any degree one, C^1-function.

Remark 5.17. The modulus may be thought of as a measure of the "electrical conductivity" for currents along $\mathbf{T}$: A "fat" torus will have a large modulus, while a very thin one will have a modulus close to zero. The modulus $m(\mathbf{T})$ is a conformal invariant: It is preserved under a conformal change of metric, since ∇ scales as length^{-1}.

Theorem 5.18 [FH2]**.** *For any divergence-free vector field ξ leaving a solid torus $\mathbf{T}$ invariant,*

$$E_{3/2}(\xi) = \int_{\mathbf{T}} \|\xi\|^{3/2}\mu \geq \frac{|\text{Flux}(\xi)|^{3/2}}{m(\mathbf{T})^{1/2}},$$

where Flux(ξ) *is the flux of the field ξ through any surface Σ representing the generator of $H_2(\mathbf{T}, \partial\mathbf{T})$; see Fig. 36.*

Proof. The theorem follows immediately from Lemma 5.15. Indeed, the Hölder inequality applied to (5.6) gives $|\text{Flux}(\xi)| \leq \|\xi\|_{L^{3/2}}\|\nabla f\|_{L^3}$; therefore

$$E_{3/2}(\xi) = (\|\xi\|_{L^{3/2}})^{3/2} \geq \frac{|\text{Flux}(\xi)|^{3/2}}{(\|\nabla f\|_{L^3})^{3/2}}.$$

The minimization over degree 1 functions f turns the L^3-norm in the denominator into the conformal modulus.   □

Remark 5.19. An incompressible diffeomorphism action preserves $\mathrm{Flux}(\xi)$, and therefore it leaves the energy of the field $h_*\xi$ bounded from below once the modulus of the torus has an upper bound. In turn, the modulus $m(\mathbf{T})$ can be bounded by purely topological quantities associated to the knot (or link) type of the torus (or of the collection of tori).

Theorem 5.20 [FH2]. *For any solid torus* $\mathbf{T}$ *of knot type K embedded in Euclidean three-space* $\mathbb{R}^3$,

$$m(\mathbf{T}) \le \frac{\sqrt{\pi}}{4(ac(K))^{3/2}}.$$

We refer to [FH2] for the proofs and for other interesting inequalities relating energy, linking, and moduli of solid tori. Conjecturally, for a nontrivial link of solid tori, $\min\{m(\mathbf{T}_1), \dots, m(\mathbf{T}_k)\}$ is majorized by a universal constant independent of k (the upper bound obtained in [FH2] is $\le \sqrt{\pi} k^{1/2}/4$).

§6. Energy of a knot

The relaxation process of magnetic tubes to a state with minimal energy raises a question on optimal embeddings of curves, or of more general submanifolds, into the space. Is there a natural way to associate such an "energy" to a submanifold so that the energy is infinite for immersions that are not embeddings, and so that the gradient flow of the energy would preserve isotopy type and evolve the submanifold to the "optimal" state?

6.A. *Energy of a charged loop*

Imagine an infinitesimal relative of a magnetic tube, a charged loop of string. Among various possible potential energies for a loop in 3-space, the one recently suggested by O'Hare [OH1] is of special interest because of its nice invariance properties (see [BFHW, FHW]).

Let $\gamma = \gamma(u)$ be a rectifiable curve embedded in $\mathbb{R}^3$, where u belongs to the circle S^1. For any pair of points $\gamma(u), \gamma(v)$ we denote by $\mathrm{dist}(\gamma(u), \gamma(v))$ the distance between them along the curve, i.e., the minimum of the lengths of the two subarcs of γ with endpoints at $\gamma(u)$ and $\gamma(v)$.

Definition 6.1 [OH1]. The *energy of the curve* γ is the following integral:

$$E(\gamma) = \iint_{S^1 \times S^1} \left\{ \frac{1}{\|\gamma(v) - \gamma(u)\|^2} - \frac{1}{|\mathrm{dist}(\gamma(v), \gamma(u))|^2} \right\} \cdot \|\dot\gamma(u)\| \cdot \|\dot\gamma(v)\| \, du dv.$$

The invariance of the energy under reparametrizations and dilations of the space is immediate.

Remark 6.2. The energy E is defined on the space of embeddings $S^1 \to \mathbb{R}^3$. It tends to infinity when the embedding becomes singular. It is a regularization of $1/r^2$-potential energy of a charged curve, while the Newton–Coulomb potential in $\mathbb{R}^3$ is $1/r$. The energies corresponding to the exponents smaller than or equal to -2 (in particular, to the case at hand) blow up as two distinct arcs of a curve get closer to each other and the curve acquires a double point. It creates an infinite barrier against any change of the knot topology. Indeed, the unregularized energy for two pieces of straight lines intersecting transversally (say, for segments of the x- and y-axes, respectively) is given by the integral

$$\iint\limits_{y\,x} \frac{dx\,dy}{(\sqrt{x^2 + y^2})^2} = \iint\limits_{\theta\,r} \frac{r}{r^2}\,dr\,d\theta,$$

which diverges at the origin.

A remarkable property of $E(\gamma)$ is a form of Möbius invariance. Recall that a *Möbius transformation* in $\mathbb{R}^3$ is a composition of a Euclidean motion, a dilation, and an inversion with respect to a sphere. Adding one point at infinity, one makes the Möbius transforms into bijections of the 3-sphere $\mathbb{R}^3 \cup \{\infty\}$.

Theorem 6.3 [BFHW, FHW]. *Let γ be a simple closed curve in $\mathbb{R}^3$ and let MT be a Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$. The following statements hold:*

  (i) *If $MT \circ \gamma \subset \mathbb{R}^3$, then $E(MT \circ \gamma) = E(\gamma)$.*
  (ii) *If $MT \circ \gamma$ passes through ∞, then $E((MT \circ \gamma) \cap \mathbb{R}^3) = E(\gamma) - 4$.*

O'Hara [OH1] proved that there exist only finitely many knot types among the curves with a given simultaneous upper bound on energy, length, and L^2-norm of the curvature. The conditions on the length and the L^2-norm of the curvature can be dropped, as the following theorem shows.

Theorem 6.4 [FHW]. *Let γ be a simple closed curve in $\mathbb{R}^3$ and let $cn(\gamma)$ (respectively, $c(\gamma, \gamma)$) denote the topological (respectively, average self-) crossing number of the knot type of γ (respectively, of the curve γ itself). Then*

$$2\pi \cdot cn(\gamma) + 4 \le E(\gamma),$$
$$12\pi \cdot c(\gamma, \gamma) \le 11 E(\gamma) + 12.$$

Notice that the average self-crossing number $c(\gamma, \gamma)$ given by the Gauss-type integral (5.1) is bounded, since the numerator undergoes a double degeneracy on the diagonal of $S^1 \times S^1$.

The energy of any round circle is $E(\text{circle}) = 4$, being the minimum of the energy for closed curves in $\mathbb{R}^3$. The theorem implies that if a closed curve satisfies

the inequality $E(\gamma) < 6\pi + 4 \approx 22.849$, then γ is unknotted (the number of crossings $cn(\gamma) \geq 3$ for any essential knot γ).

Using the exponential upper bound of the number of distinct knots with a given bound for the number of crossings, one obtains the following

Corollary 6.5 [FHW]. *The number of (the isomorphism classes of) knots that can be represented by curves whose energy E does not exceed N is bounded by*

$$2 \cdot (24^{-4/2\pi}) \cdot (24^{1/2\pi})^N \approx (0.264)(1.658)^N.$$

Milnor [Mil1] showed that for the *total curvature*

$$TK(\gamma) = \int \left| \left(\frac{\dot{\gamma}(u)}{\|\dot{\gamma}(u)\|} \right)' \right| du$$

(where $'$ and $\cdot$ stand for the derivative in u), the inequality $TK(\gamma) \leq 4\pi$ implies that γ is unknotted ($TK(\text{circle}) = 2\pi$). However, for any given $\epsilon > 0$ there exist infinitely many knot types having representatives of total curvature $TK \leq 4\pi + \epsilon$.

Remarks 6.6. The total energy can be similarly assigned to a link $(\gamma_1, \ldots, \gamma_k)$, which consists of k disjoint embeddings of S^1 to $\mathbb{R}^3$:

$$TE(\gamma_1, \ldots, \gamma_k) = \sum_{i=1}^{k} E(\gamma_i, \gamma_i) + \frac{1}{2} \sum_{i,j=1,i\neq j}^{k} E(\gamma_i, \gamma_j),$$

where $E(\gamma_i, \gamma_i) = E(\gamma_i)$, and for $i \neq j$,

$$E(\gamma_i, \gamma_j) = \iint\limits_{S^1 \times S^1} \frac{\|\dot{\gamma}_i(u)\| \cdot \|\dot{\gamma}_j(u)\|}{\|\gamma_i(u) - \gamma_j(u)\|^2} \, du dv.$$

Given $N > 0$ there are finitely many link types that have representatives with $TE \leq N$ (see [FHW]).

Remarks 6.7. For a divergence-free field confined to nontrivially knotted or linked tubes there is a lower bound of the magnetic energy, as discussed in Section 2.B. Moffatt [Mof5] suggested using these lower bounds of the energy as the invariants of (the tubular neighborhoods of) knots and links.

Namely, for any knot, consider a satellite flux-tube of volume *vol* carrying an "untwisted" vector field ξ of flux Flux (across any meridian section of the tube) and look at the associated energy of this vector field. This energy can be decreased by a diffeomorphism action, preserving both *vol* and Flux, to a topological accessible minimum. On dimensional grounds, this minimal energy $E(\xi) = m \cdot (\text{Flux})^2(vol)^{-1/3}$, where $m = m(\text{Flux}, vol)$ is a positive real number depending on the knot topology. If for a given knot, different local minima of the energy exist, then the sequence $\{m_0, m_1, \ldots, m_r\}$ of possible values could be reasonably described as the energy spectrum of the knot (neighborhood). The lowest

number m_0 provides a possible natural measure of the knot complexity (see also [C-M]).

It would be interesting to relate the final positions of the vortex magnetic tubes under the E_2-energy relaxation to the shape of the curves, realizing the minimum of an appropriate energy function on curves. The critical points of such an energy would correspond to the equilibrium states for the Moffatt spectrum.

6.B. Generalizations of the knot energy

There is a variety of Möbius invariant generalizations of the knot energy (see, e.g., [D-S, AuS, KuS]). Imagine a charge uniformly spread over a k-dimensional submanifold $M \subset \mathbb{R}^n$.

Definition 6.8. Given a function f, define the f-*energy* to be

$$E_f(M) = \iint_{M \times M} \frac{f(x, y)}{\|x - y\|^{2k}} \, d \operatorname{vol}_M(x) \, d \operatorname{vol}_M(y).$$

Regard the function f on $M \times M$ as a function of three arguments, $f = f(M, x, y)$.

Definition 6.9. A function $f(M, x, y)$ is g-*invariant* under the action of a map $g : M \to M$ if $f(g_* M, g(x), g(y)) = f(M, x, y)$.

Theorem 6.10 [D-S, AuS, KuS]. *Any scale and Möbius invariant factor f gives rise to the energy E_f invariant with respect to the Möbius transformations of $\mathbb{R}^n \cup \{\infty\}$.*

The scale invariance of the integrand justifies the choice of the power $2k$ rather than the physically meaningful $n - 2$ in the denominator of the energy in $\mathbb{R}^n$. The submanifold $M \subset \mathbb{R}^n \cup \{\infty\}$ can be viewed as a submanifold of $S^n \subset \mathbb{R}^{n+1}$ via the stereographic projection. Such a projection extends to a Möbius transformation of $\mathbb{R}^{n+1}$, while the energy formula does not depend on the ambient dimension.

Proof. The statement follows from the Möbius invariance of the integrand with $f \equiv 1$. For the latter case the scale invariance is evident, while the invariance under inversion $r \mapsto \tilde{r} := r/\|r\|^2$ follows from Fig. 37.

The similar triangles Oxy and $O\tilde{x}\tilde{y}$ provide the identity

$$\frac{\|\tilde{x}\| \cdot \|\tilde{y}\|}{\|\tilde{x} - \tilde{y}\|^2} = \frac{\|x\| \cdot \|y\|}{\|x - y\|^2}.$$

On the other hand, the inversion transforming M into $\tilde{M}$ expands conformally the lengths at x by the factor $\|\tilde{x}\|/\|x\|$. The corresponding change of the volume
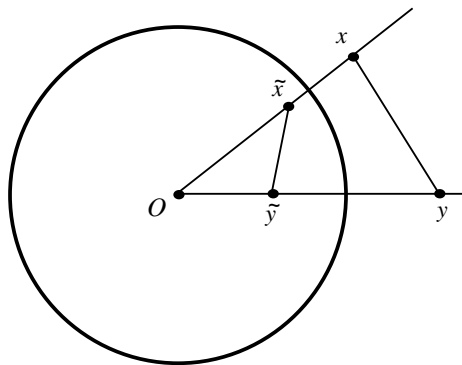
FIGURE 37. Triangles Oxy and $O\tilde{x}\tilde{y}$ are similar (where $\tilde{x}$, $\tilde{y}$ are the inverses of x, y).

element is

$$d \operatorname{vol}_{\tilde{M}}(\tilde{x}) = (\|\tilde{x}\|/\|x\|)^k \, d \operatorname{vol}_M(x).$$

This shows that the integrand, as a whole, remains invariant under inversion, and hence under an arbitrary Möbius transformation.   □

If $f \equiv 1$, the integrand blows up as x approaches y, and therefore the energy is infinite for any M. The regularizing factor f is designed to compensate the singularity, and so it vanishes as $x \to y$.

The list of properties desired from a particular regularization usually includes the infinite barrier against self-crossings, the Möbius invariance, and boundedness of the energy from below. More restrictive is the property of approximate additivity for the connected sum of two remote knots, and the requirement that the energy contribution of any two disjoint arcs would be independent of whether they are in the same component of the link (see the discussion in [AuS]).

To give an example, return to the case of a knot $\gamma \in \mathbb{R}^3$. Define a specific regularization $f_0 : \gamma \times \gamma \to \mathbb{R}$ by the following construction.

Definition 6.11 [D-S]. Given a point $x \in \gamma$ and any other point $p \in \mathbb{R}^3$, there is a unique circumference (or straight line) $S_x(p)$ tangent to γ and passing through p. Thus given two points x and y of γ, we have two oriented circumferences $S_x(y)$ and $S_y(x)$ that meet at equal angles at x and y. Let α be the *angle* at which these two circles meet in $\mathbb{R}^3$. These circles and, in particular, the angle α are defined in a Möbius invariant manner. Set the *special weight* f_0 to be the function $f_0 := 1 - \cos \alpha$. (The angle α can also be defined in the case of an arbitrary k-dimensional submanifold in $\mathbb{R}^n$ by replacing the circumferences $S_x(y)$ by k-spheres [KuS]).

Proposition 6.12 [D-S]. *The knot energy E_{f_0} defined by the special weight f_0 is equivalent to O'Hare's energy E modulo a constant: If γ is a closed curve in $\mathbb{R}^3$,*

then

$$E_{f_0}(\gamma) = E(\gamma) - 4.$$

It was shown in [FHW] that for an irreducible knot there is a representative having minimal energy among all simple loops of the same knot type. A criterion describing the "optimal" (minimizing the energy) states was obtained in [OH2]. We also refer to the paper [KuS] for nice stereo-pairs of optimal links (with the number of crossings $cn \leq 8$) that allow one to visualize the three-dimensional picture.

Remark 6.13. Note that the number of critical points of a function on the space of embeddings $S^1 \to \mathbb{R}^3$ can be minorized by Morse theory and by Vassiliev's calculation of the Betti numbers of the space of embeddings [VasV].

Unlike knots, plane curves (immersions $S^1 \to \mathbb{R}^2$) generically have self-intersection points. The simplest singular plane curves (forming the discriminant hypersurface in the space of maps $S^1 \to \mathbb{R}^2$) have either a triple point or a point of self-tangency (see Fig. 38). A treatment of the corresponding theory of Vassiliev-type invariants for the plane and Legendrian curves can be found in [Arn21, Aic, L-W, Vir, Pl1, 2, PlV, Shm, Tab2, Gor, FuT].
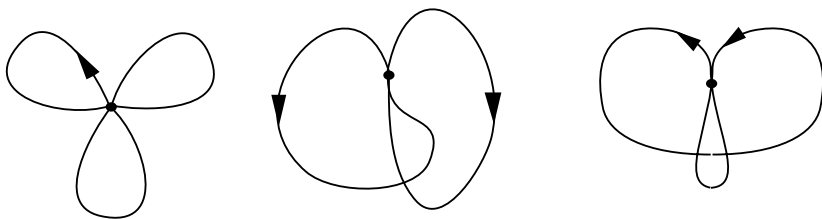
FIGURE 38.   Plane curves with triple points and self-tangencies.

Problems 6.14. (A) Is there an energy functional on the space of immersions that is infinite on the discriminant and possesses the property of Möbius invariance (and/or other properties from the discussion above)? Conjecturally, there will be only finitely many homotopy classes of immersed curves whose would-be energy is bounded from above.

(B) Are there asymptotic generalizations of invariants of plane curves similar to those discussed above for the linking of space curves? We refer to [Aic, L-W] for very suggestive integral formulas of the invariants.

Remark 6.15 (D. Kazhdan). The growth rate of the number of types of immersions into the plane as a function the crossing number suggests the existence of a negative curvature metric in the corresponding spaces of immersions.

§7. Generalized helicities and linking numbers

This section describes various generalizations of the helicity integral to manifolds with boundary, to the non-simply connected and higher-dimensional cases, as well as to magnetic tubes forming links detected by certain higher-order link invariants.

7.A. *Relative helicity*

The helicity of a vector field in a simply connected manifold with boundary (say, in a domain of $\mathbb{R}^3$) is well-defined, provided only that the field is tangent to the boundary. A vector field crossing the boundary possesses neither the ergodic version of the definition (some of its trajectories leave the region, and therefore their asymptotic linking cannot be specified) nor the integral one (the formula has to include a boundary term). However, the vector fields identical outside the region can be compared by means of the *relative linking* of their trajectories in the interior [Ful, B-F].

The definition of the relative linking number for nonclosed curves rests on the introduction of "reference arcs" with the same endpoints and closing up the curves; Fig. 39 (see [Ful], where this construction is applied to the study of DNA knottedness).
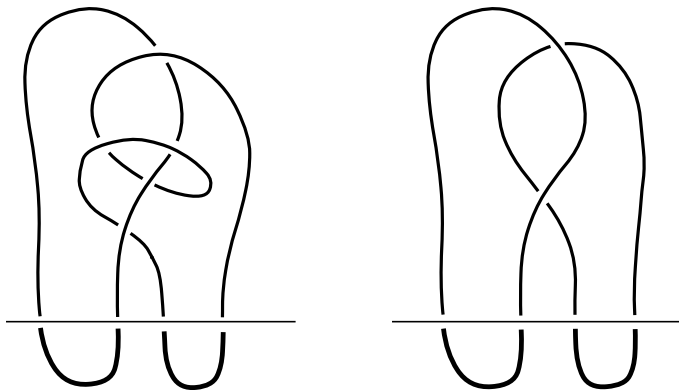
FIGURE 39.   Nonclosed curves have relative linking with respect to arcs outside the region.

The continuous version is as follows [B-F]. Suppose that a domain in the space $\mathbb{R}^3$ (or a closed simply connected manifold M^3) is split into two simply connected regions A and B separated by a boundary surface S. Assume further that two divergence-free vector fields ξ and η in A coincide on the boundary S and have the same extension ζ into the region B. Call the extended fields in M respectively $\tilde{\xi}$ and $\tilde{\eta}$. Abusing notation we will denote them as the sums $\tilde{\xi} = \xi + \zeta$ and $\tilde{\eta} = \eta + \zeta$ (where ξ, η, ζ are regarded as the (discontinuous) vector fields in the entire manifold M with supports $\operatorname{supp} \xi$, $\operatorname{supp} \eta \subset A$, and $\operatorname{supp} \zeta \subset B$).

Lemma–definition 7.1. *The difference of the helicities of the fields $\tilde{\xi}$ and $\tilde{\eta}$*

$$\Delta\mathcal{H} = \mathcal{H}(\tilde{\xi}) - \mathcal{H}(\tilde{\eta})$$

is independent of their common extension ζ in the region B, and hence it measures the relative helicity *of the fields ξ and η in A.*

Proof. Define the (closed) two-forms α, β, and ω (by substituting the vector fields ξ, η, and ζ with respect to the volume form μ on M: $i_\xi\mu = \alpha$, etc.). Then one has to show that the difference

$$\mathcal{H}(\tilde{\xi}) - \mathcal{H}(\tilde{\eta}) := \int_M (\alpha + \omega) \wedge d^{-1}(\alpha + \omega) - \int_M (\beta + \omega) \wedge d^{-1}(\beta + \omega)$$

does not depend on ω. One readily obtains

$$\Delta\mathcal{H} = \int_M \alpha \wedge d^{-1}\alpha - \int_M \beta \wedge d^{-1}\beta + \int_M (\alpha - \beta) \wedge d^{-1}\omega + \int_M \omega \wedge d^{-1}(\alpha - \beta).$$

Here d^{-1} applied to a discontinuous 2-form is a continuous 1-form (the "form-potential"). The terms in $\Delta\mathcal{H}$ containing ω are $\int_M(\alpha - \beta) \wedge d^{-1}\omega + \int_M \omega \wedge d^{-1}(\alpha - \beta)$, and we want to show that their contribution vanishes.

Integrating by parts one of the terms, we come to $2\int_M(\alpha - \beta) \wedge d^{-1}\omega$, which, in turn, is equal to $2\int_A(\alpha - \beta) \wedge d^{-1}\omega$, since $\mathrm{supp}(\alpha - \beta) \subset A$.

On the other hand, in A the 1-form $d^{-1}\omega$ is the differential of a function: $d^{-1}\omega = dh$. Indeed, it is closed (the differential $d(d^{-1}\omega) = \omega$ vanishes in A due to the condition on $\mathrm{supp}\,\zeta = \mathrm{supp}\,\omega \subset B$), and hence it is exact in the simply connected region. Hence,

$$2\int_A(\alpha - \beta) \wedge d^{-1}\omega = 2\int_A(\alpha - \beta) \wedge dh = 2\int_S h(\alpha - \beta) = 0,$$

where the last equality is due to the assumption on the identity of the fields ξ and η on the boundary S. This proves that $\Delta\mathcal{H}$ is not affected by the choice of the extension ζ. $\qquad\square$

The relative helicity of a field transversal somewhere to the boundary of M is no longer invariant under the action of volume-preserving diffeomorphisms of M.

Remark 7.2. The phenomenon of the same type holds for divergence-free vector fields on non-simply connected manifolds. A true linking number does not exist for such a case, but two "homologically equivalent" fields can be compared with each other. A nice application to the *linking numbers for cascades* can be found in [GST].

Choose a nonsingular C^1 vector field inside a solid torus such that the flow lines are transversal to its 2-disks, as in Fig. 40a. In this setting one can define the following version of the linking number. We fix the direct product structure $S^1 \times D^2$ in the solid torus trivializing its fibration over S^1.

The *topological linking* of two long pieces of orbits is the algebraic number of times one trajectory winds around the other. Namely, the projections of the orbits to the disk form a moving pair of points in the same 2-disk. The linking number is the rotation number of one point around the other. This definition extends to the case of the cascades of periodic orbits in a solid torus. A cascade flow in the solid torus cyclically interchanges smaller invariant disks in the transverse section and repeats itself inside these disks (Fig. 40b).
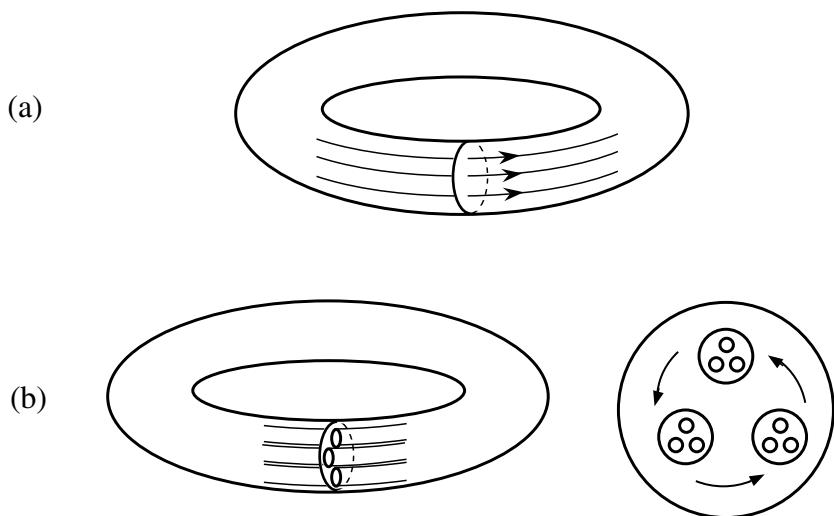
FIGURE 40.   (a) A solid torus with a vector field transversal to the 2-disk D^2. (b) Cascade of embedded solitori.

On the other hand, to the piece $0 \leq t \leq T$ of a single orbit of a C^1 flow one can associate the *(infinitesimal) self-linking number* by counting how many times a tangent vector in the disk direction turns around the orbit. For almost all points, the infinitesimal self-linking number has a limit as $T \to \infty$, and this limit can be described by a spatial integral of the appropriate derivative [Rue].

Gambaudo, Sullivan, and Tresser showed in [GST] that the sequence of the topologically defined average linking numbers between successive orbits in the cascade converges to the average self-linking number of the invariant set. They also described the sequences of rational numbers (in a sense, counterparts of the rotation numbers of maps of a circle into itself) that can appear as the average linking numbers in a cascade of iterated torus knots.

7.B. *Ergodic meaning of higher-dimensional helicity integrals*

The higher-dimensional integrals generalizing the helicity of a vector field in $\mathbb{R}^3$ were introduced by Novikov [Nov1]. His idea was to extend to closed differential forms on higher-dimensional spheres (which are not necessarily the pullbacks of

the forms from the spheres of smaller dimension) the Whitehead operations in the homotopy groups of the spheres (simulating the approach, transforming the Hopf invariant on the homotopy group $\pi_3(S^2)$ into the helicity of divergence-free vector fields on S^3).

An ergodic interpretation of Novikov's constructions encounters the following difficulty. Unlike the three-dimensional case, where the asymptotic linking number is defined for almost every pair of trajectories, the field lines are not linked if the dimension of the ambient manifold is greater than 3. Thus, instead of the curves, one should consider the submanifolds of higher dimensions. But for nonclosed submanifolds of dimension ≥ 2 one lacks a satisfactory generalization of the system of short paths.

We consider the geometric meaning of the invariants of closed two-forms on manifolds of arbitrary dimension. For odd-dimensional manifolds quantities like $\int d^{-1}\alpha \wedge \beta \wedge \cdots \wedge \omega$ arise as first integrals in the theory of an ideal or barotropic fluid (Sections I.9, VI.2) or in the Chern–Simons theory (Section 8.A). Here the asymptotic linking number of every pair of field lines is replaced by the linking of a trajectory with a foliation of codimension 2. For even-dimensional manifolds the Novikov invariants are described as the average *nongeneric* linkings [Kh1]. The interpretation presented here is an ergodic counterpart of the Poincaré duality that translates facts on the differential forms into a description of the intersections of their kernel foliations (cf. Remark 4.7).

Let M^n be a compact connected manifold without boundary and $H_1(M, \mathbb{R}) = H_2(M, \mathbb{R}) = 0$. Denote closed (and hence, exact) two-forms on M by $\alpha, \beta, \ldots \in \Omega^2(M)$, while $d^{-1}\alpha, d^{-1}\beta, \ldots \in \Omega^1(M)$ are arbitrary primitive one-forms (form-potentials) for the corresponding two-forms. We start with the following simple observations:

Proposition 7.3. *(i) For an odd-dimensional manifold M^{2m+1} and arbitrary $m+1$ closed two-forms $\alpha, \beta, \ldots, \omega$, the Hopf-type integral $I(\alpha, \beta, \ldots, \omega) = \int_M d^{-1}\alpha \wedge \beta \wedge \cdots \wedge \omega$ is symmetric under the permutations of $\alpha, \ldots, \omega$ and does not depend on the choice of the primitive $d^{-1}\alpha$.*

(ii) [Nov1] On a four-dimensional manifold M^4 for any two 2-forms α and β that obey the conditions $\alpha \wedge \alpha = \beta \wedge \beta = \alpha \wedge \beta = 0$, the integrals

$$J_1(\alpha, \beta) = \int_M d^{-1}\alpha \wedge \alpha \wedge d^{-1}\beta \qquad and$$

$$J_2(\alpha, \beta) = \int_M d^{-1}\alpha \wedge \beta \wedge d^{-1}\beta$$

do not depend on the choices of $d^{-1}\alpha$ and $d^{-1}\beta$.

In [Nov1], Novikov defined a set of invariants on manifolds of an arbitrary dimension, and we consider the case of M^4 for illustration. We are going to represent these integrals as the generalized linking numbers of certain foliations associated to the differential forms.

Definition 7.4. A closed 2-form α of rank ≤ 2 on a manifold M^n determines a (singular) foliation (called a *kernel foliation*) of codimension 2 in M: the tangent plane to this foliation at any point of M is spanned by the $(n-2)$-vector being the kernel of α at that point.

If the manifold is equipped with a volume form μ, then this foliation is generated by the field of $(n-2)$-vectors $\mathcal{A}$ whose substitution $i_{\mathcal{A}}$ into the volume form gives α (i.e., $i_{\mathcal{A}}\mu = \alpha$).

Proposition 7.5. *The kernel field of a closed two-form is completely integrable. In particular, for the form of rank ≤ 2, it spans a foliation of codimension ≥ 2.*

Proof is an application of the Frobenius integrability criterion. $\square$

Remark 7.6. Without the restriction on rank of the two-form α the corresponding kernel $(n-2)$-vector field $\mathcal{A}$ is generically indecomposable. The conditions $\alpha \wedge \alpha = \beta \wedge \beta = 0$ on the pair α, β in ii) in the above proposition are exactly the limitations on the ranks: $rk(\alpha), rk(\beta) \leq 2$. The third condition $\alpha \wedge \beta = 0$ ensures that the kernel foliations (of dimension 2 in M^4) determined by the forms α and β (near a point where $rk(\alpha) = rk(\beta) = 2$) are allocated in the following peculiar way. The intersections of their leaves form a 1-dimensional foliation, provided that α and β are not proportional. Moreover, the distribution spanned by the kernels of α and β determines in this case a 3-dimensional foliation [Arn9].

Definition 7.7. The *average linking* of a curve Γ with the foliation $\mathcal{A}$ is the flux of the two-form $\alpha = i_{\mathcal{A}}\mu$ through an arbitrary surface $\partial^{-1}\Gamma$ bounded by Γ:

$$lk(\Gamma, \mathcal{A}) = \int_{\partial^{-1}\Gamma} \alpha = \int_{\Gamma} d^{-1}\alpha.$$

The following proposition motivates the definition of $lk(\Gamma, \mathcal{A})$.

Proposition 7.8. *The number $lk(\Gamma, \mathcal{A})$ coincides with the average linking number (evaluated with the help of the linking form $G \in \Omega^{n-2}(M) \times \Omega^1(M)$) of the leaves of foliation $\mathcal{A}$ with the curve Γ.*

Proof. By definition of the form G the linking number of two submanifolds P and Q in M is given by the integral $\iint_{P \times Q \subset M \times M} G$, see Section 4. Therefore,

$$\iint_{\mathcal{A} \times \Gamma} G = \iint_{M \times \Gamma} i_{\mathcal{A}} G \wedge \mu = \iint_{M \times \Gamma} G \wedge i_{\mathcal{A}}\mu = \iint_{M \times \Gamma} G \wedge \alpha = \int_{\Gamma} d^{-1}\alpha.$$

(Here the first identity is the definition of $\iint_{\mathcal{A} \times \Gamma} G$, the last one is the main property of G: the operator corresponding to the linking form acts on the exact differential 2-forms just like the operator d^{-1}; see Section 4.D.) $\square$

By analogy with the three-dimensional case, we can now define an *asymptotic linking* $lk_\xi(x, \mathcal{A})$ of the trajectory of a vector field ξ passing through a point $x \in M$ with the foliation $\mathcal{A}$. It is the time-average of the linking number with $\mathcal{A}$ of the curve $\Gamma_T(x)$ consisting of the long segment (for time $0 \le t \le T$) of the ξ-trajectory $g_\xi^t x$ starting at $x \in M$ and of a short closing path:

$$lk_\xi(x, \mathcal{A}) = \lim_{T \to \infty} \frac{1}{T} lk(\Gamma_T(x), \mathcal{A}).$$

Definition 7.9. The *average linking number* of the vector field ξ with the foliation $\mathcal{A}$ defined on the manifold M equipped with the volume form μ is

$$lk_\xi(\mathcal{A}) = \int_M lk_\xi(x, \mathcal{A})\mu.$$

Theorem 7.10. *Let* $\alpha, \beta, \ldots, \omega$ *be a set of* $m + 1$ *closed two-forms on* M^{2m+1}. *Assume that the rank of one of the forms (for example, α) is at most* 2. *Then the Hopf-type integral* $I(\alpha, \beta, \ldots, \omega) = \int_M d^{-1}\alpha \wedge \cdots \wedge \omega$ *coincides with the average linking number of the vector field* ξ *with the foliation* $\mathcal{A}$:

$$I(\alpha, \ldots, \omega) = lk_\xi(\mathcal{A}),$$

where the fields ξ *and* $\mathcal{A}$ *are defined by* $i_\xi\mu = \beta \wedge \ldots \wedge \omega$ *and* $i_\mathcal{A}\mu = \alpha$.

Proof. The proof is a straightforward application of the Birkhoff ergodic theorem.
$\square$

The rank of α is essential merely to define the foliation $\mathcal{A}$. In the general case, we would consider a linking with an abstract $(n - 2)$-vector field instead of an $(n - 2)$-dimensional foliation. If, conversely, all these forms have rank ≤ 2 (of course, this is seldom the case), then one can interpret the number $I(\alpha, \ldots, \omega)$ as the multilinking of all the corresponding foliations.

Namely, the usual linking number is a bilinear form on the space of disjoint submanifolds of appropriate dimensions: It is defined for a pair of submanifolds P^k and Q^l in M^n, subject to the conditions $k + l = n - 1$ and $P \cap Q = \emptyset$. Similarly, we define the multilinking number as a multilinear form on the space of r-tuples of submanifolds $(P_1, \ldots, P_r)$ such that

$$(7.1) \qquad \sum_{i=1}^{r} \text{codim } P_i = n + 1$$

and

$$(7.2) \qquad \bigcap_{i=1}^{r} P_i = \emptyset.$$

Definition 7.11. The *mutual linking number* of r oriented closed submanifolds $P_1, \ldots, P_r$ in $M = \mathbb{R}^n$ (or S^n) satisfying the condition above is the signed number

of the intersection points of a manifold $F \subset M$, bounded by one of these surfaces $P_i = \partial F$, with the intersection of all the other submanifolds.

If these submanifolds are equipped with some transversal orientations, then so are all the manifolds bounded by them and all their intersections, and hence the signs of the intersection points are well-defined. For example, it is possible to link three circles in the plane or two spheres and one circle in 3-space (Fig. 41).
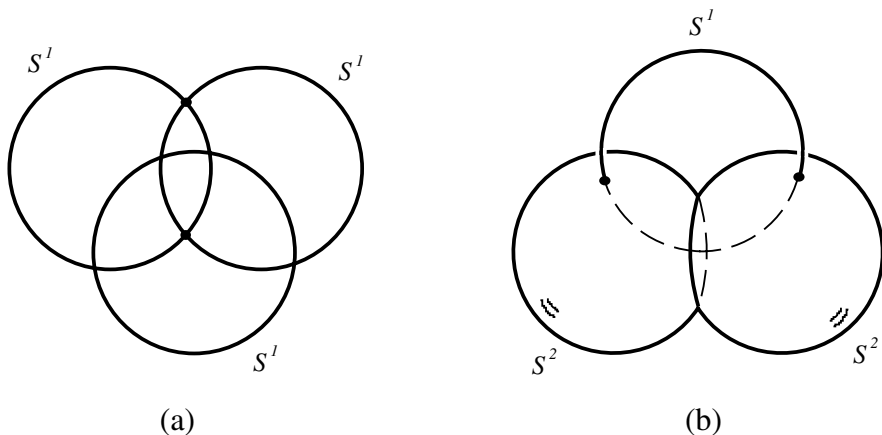

FIGURE 41. Links of (a) three circles in the plane; (b) two spheres and a circle in 3-space.

Note that the mutual linking number of a collection $P_1, \dots, P_r \subset M$ is the usual linking number of the submanifold $P_1 \times \cdots \times P_r \subset M \times \cdots \times M$ with the diagonal $\Delta = \{(x, \dots, x) \mid x \in M\} \subset M \times \cdots \times M$.

We recall that every closed 2-form of rank ≤ 2 determines a foliation of codimension 2. If the leaves were compact, one could consider the mutual linking of these leaves for $(m + 1)$ two-forms in M^{2m+1} due to

$$\sum_{i=1}^{m+1} (\text{codimension of leaves}) = 2m + 2 = \dim M + 1.$$

So in these terms, Theorem 7.10 above reads as

Theorem 7.10′. *The Hopf-type invariant is equal to the average asymptotic multilinking number of the leaves determined by the given 2-forms.*

To describe the ergodic meaning of the Novikov integrals J_1 and J_2, we shall extend the concept of multilinking. We are going to drop the codimension condition (7.1) if it is compensated in (7.2) by an assumption on the nongeneric intersection of the submanifolds. For example, two circles S^1 and a sphere S^2 cannot be linked in $\mathbb{R}^3$ (one can untie any configuration of them not passing through any triple

point; Fig. 42a). However, if these two circles are two meridians of the same ball (and so their intersection S^0 consists of two points), the linking may be nontrivial (Fig. 42b). Namely, one cannot remove S^2 far from the two meridians unless it passes through an intersection point of these two meridians.
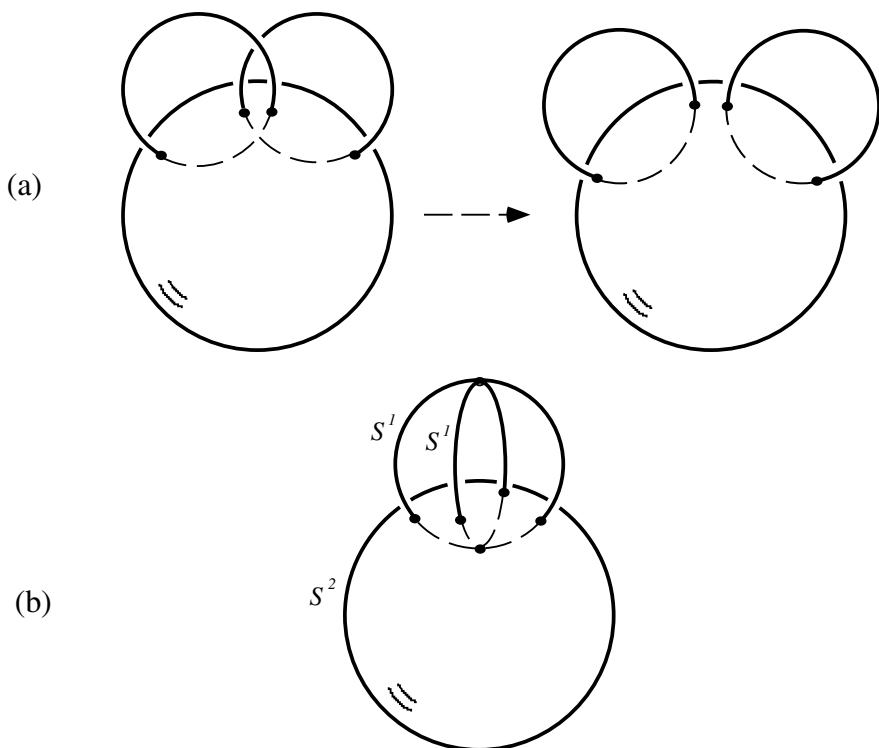
FIGURE 42.   (a) Generic and (b) nongeneric linkings of two circles and a sphere.

In the definition of invariants J_i, the $(\alpha \wedge \beta = 0)$-type conditions provide the nongeneric intersections of the corresponding leaves.

Theorem 7.12. *The invariant $J_1(\alpha, \beta)$ (respectively, $J_2(\alpha, \beta)$) coincides with the average linking number of the foliation $\mathcal{A}$ of the 2-form α (respectively, of the foliation $\mathcal{B}$ of β) with the vector field ξ satisfying $i_\xi \mu = d(d^{-1}\alpha \wedge d^{-1}\beta)$.*

Roughly speaking, each of these two amounts is the average linking number of the 1-dimensional foliation formed by the intersections of $\mathcal{A}$ and $\mathcal{B}$ with the foliation $\mathcal{A}$ or $\mathcal{B}$ (determined, respectively, by α or by β).

Remark 7.13. The Hopf-type invariants arise in [Nov1] in a context of quantum anomalies. Consider the space $\mathcal{L}$ of smooth mappings $f : S^q \to M^n$ homotopic

to zero. To a closed $(q + 1)$-form θ on M one naturally associates a multivalued function $F_\theta(f)$ (or a closed 1-form δF_θ) on the space $\mathcal{L}$:

$$F_\theta(f) = \int_{f(D^{q+1})} \theta.$$

Here $f : S^q \rightarrow M^n$ is extended to a mapping $D^{q+1} \rightarrow M^n$ of the ball D^{q+1} bounded by the sphere. Closedness of the $(q + 1)$-form θ implies that δF depends just on $f\big|_{\partial D^{q+1}=S^q}$.

The differential δF of a multivalued functional $F(f)$ on the space $\mathcal{L}$ is said to be *local* if it depends on f and on a finite number of its derivatives. For $n \geq q + 1$ all multivalued functionals $F(f)$ with local differentials are the sums of a local univalued functional and $F_\theta(f)$ [Nov2]. A construction of multivalued functionals for $n \leq q + 1$ that conjecturally describes all functionals with local differentials is given in [Nov1].

There is an integer lattice inside the space of θ's consisting of homotopy invariant elements. The meaning of this lattice is exactly equivalent to the role of the usual integer-valued Hopf invariant of mappings $S^3 \rightarrow S^2$ among all asymptotic linking invariants for arbitrary divergence-free vector fields on S^3. It is natural to call the appearance of the integral lattice a quantization condition [Nov1].

7.C. Higher-order linking integrals

The Gauss linking integral fails to detect the entanglements of curves in $\mathbb{R}^3$ with an equal number of "oppositely signed crossings." The Whitehead link and the Borromean rings are examples of this kind (see Fig. 43). In this section we consider the higher-order invariants called Massey numbers (see [Mas]) that generalize the linking number of two curves and allow one to detect more general curve configurations.
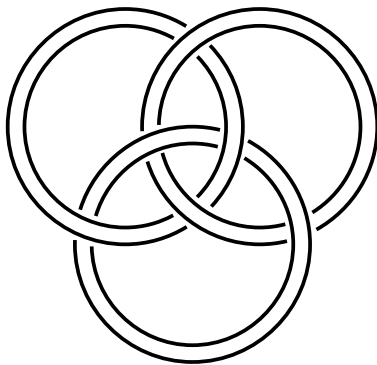

FIGURE 43.   Three solid tori form the Borromean rings.

The formalism of differential forms for the hierarchy of higher link invariants was developed in [Mas] (see also [MRe]). This notion was introduced in a magnetohydrodynamical setting in the paper [MSa] and rediscovered in [Be1, E-B], to which we refer for more detail (cf. [LS2]). The topological obstruction rules for the links in nematics and in certain superfluids can be found in [MRe].

The helicity of field tubes is quadratic in the magnetic fluxes (see formula (2.1)), and therefore it describes a *second-order* invariant. For the Borromean rings the Gauss integral taken over any two rings vanishes and so does the helicity of the entire tube configuration. The Borromean rings can be distinguished from the three totally unlinked rings by means of a *third-order* linking invariant, cubic in the fluxes.

We start with the three closed curves forming the Borromean rings and encased in toroidal volumes $\mathbf{T}_k$, $k = 1, 2, 3$. The field ξ_k is concentrated in the tube $\mathbf{T}_k$, vanishes outside, and has unit flux in $\mathbf{T}_k$. Denote by A_k a vector-potential for ξ_k and by ϕ_k the associated 1-form-potential. (In invariant terms, one first finds a closed two-form $\alpha_k = i_{\xi_k}\mu$, which is the substitution of the field ξ_k into the volume form μ, and then $\phi_k = d^{-1}\alpha_k$ is any primitive one-form such that $d\phi_k = \alpha_k$.)

Having defined the two-forms $\omega_{ij} = \phi_i \wedge \phi_j = -\phi_j \wedge \phi_i$ for $i \neq j$ (note: $d\omega_{ij} = 0$ outside of $\mathbf{T}_i \cup \mathbf{T}_j$), the helicity integral becomes

$$\mathcal{H}_{ij} := \mathcal{H}(\xi_i, \xi_j) = \int_{\mathbf{T}_i \cup \mathbf{T}_j} \alpha_i \wedge d^{-1}\alpha_j = \int_{\mathbf{T}_i} \alpha_i \wedge \phi_j = \int_{\mathbf{T}_i} d\omega_{ij}$$

due to $\operatorname{supp}\alpha_i \subset \mathbf{T}_i$. By virtue of the Stokes formula, the latter integral is equal to $\mathcal{H}_{ij} = \int_{\partial \mathbf{T}_i} \omega_{ij}$. All the quantities $\mathcal{H}_{12}, \mathcal{H}_{23}, \mathcal{H}_{31}$ vanish for the Borromean rings.

One can modify the form ω_{ij} inside the tubes to make it closed everywhere. Namely, one has to add the 2-form $h_{(j)i} \cdot \alpha_j$ to ω_{ij} inside $\mathbf{T}_j$ and to subtract $h_{(i)j} \cdot \alpha_i$ from ω_{ij} inside $\mathbf{T}_i$, where $h_{(i)j}$ is a scalar potential satisfying $\phi_j = dh_{(i)j}$. The function $h_{(i)j}$ exists in the tube $\mathbf{T}_i$ (but not globally), since the magnetic field ξ_j (and the corresponding two-form $\alpha_j = d\phi_j$) is zero there, and because $\mathbf{T}_i$ is not linked with $\mathbf{T}_j$. The Poincaré lemma applied to the new ω_{ij} guarantees that there is a one-form θ_{ij} such that $\omega_{ij} = d\theta_{ij}$.

Definition 7.14. The *third-order linking integral* is

$$\mathcal{H}_{ijk} = \int_{\partial \mathbf{T}_i} \omega_{ijk} = \int_{\mathbf{T}_i} d\omega_{ijk} = -\int_{\partial \mathbf{T}_k} \omega_{ijk}$$

for distinct i, j, k, where ω_{ijk} is the *Massey triple product*

$$\omega_{ijk} = \phi_i \wedge \theta_{jk} + \theta_{ij} \wedge \phi_k.$$

As a matter of fact, the Massey product is a map defined on cohomology classes. This implies both the gauge invariance of the third-order linking integral $\mathcal{H}_{ijk}$ and its invariance under deformations of the three curves. It vanishes for three unlinked circuits but is equal to ± 1 for the Borromean rings.

Remark 7.15. In the language of vector calculus the Massey product becomes

$$\Omega_{ijk} = A_i \times \text{curl}^{-1}\, \Omega_{jk} + \text{curl}^{-1}\, \Omega_{ij} \times A_k,$$

where Ω_{ij} is the vector field $A_i \times A_j$ modified inside the tubes to make it divergence free and hence to provide the existence of a potential $\text{curl}^{-1}\, \Omega_{ij}$ (see [E-B]).

The purely cohomological description of the numbers $\mathcal{H}_{ijk}$ is as follows (see, e.g., [MRe]). Let the curves $\Gamma_k, k = 1, 2, 3$, constitute the "axes" of the Borromean rings $\mathbf{T}_k$ in S^3. A closed 1-form ϕ_k is the Alexander dual of the circle Γ_k. It is defined in $S^3 \setminus \Gamma_k$ and can be regarded as a linking form: For any closed curve γ in this complement $\int_\gamma \phi_k = lk(\Gamma_k, \gamma)$.

The condition $lk(\Gamma_i, \Gamma_j) = 0$ allows one to find a 1-form ω_{ij} on $S^3 \setminus (\Gamma_i \cup \Gamma_j)$ such that $d\omega_{ij} = \phi_i \wedge \phi_j$. Now $\omega_{123} = \omega_{12} \wedge \phi_3 + \phi_1 \wedge \omega_{23}$ is defined on S^3 with the three circles removed, and it can be integrated over the boundary $\partial \mathbf{T}_1$.

Remark 7.16. This is the starting point for a hierarchy of the invariants. (The invariants of order n can be defined for configurations whose invariants of order $\leq n-1$ vanish.)

A fourth-order linking invariant capturing the Whitehead link was suggested in [A-R]. Consider Seifert surfaces corresponding to two closed disjoint curves. For each of the curves such a surface can be chosen not to intersect the other curve, provided that the linking number of the pair vanishes. Then, generically, the intersection of the two Siefert surfaces is a closed curve equipped with a framing. The self-linking number of the framed curve is a topological invariant, and it is independent of the choice of the surfaces [Sat]. By making the curves into thin solid tori, one can obtain an integral form of the invariant [A-R].

Remark 7.17. Another way to generalize the linking number to more complicated links was suggested by Milnor [Mil2]. For all necessary definitions of *higher-order Milnor coefficients* and for their relation to the higher-order Massey linking numbers see [Mil2, Tu1, Por, MRe].

Remark 7.18. In all the constructions of this section, the magnetic field is assumed to be highly degenerate: It is concentrated in toroidal tubes with all the trajectories closed inside the tubes. Such fields form a slim set of infinite codimension in the space of all divergence-free vector fields. No asymptotic version of these constructions is known.

The dream is to define such a hierarchy of invariants for generic vector fields such that, whereas all the invariants of order $\leq k$ have zero value for a given field and there exists a nonzero invariant of order $k+1$, this nonzero invariant provides a lower bound for the field energy.

Remark 7.19. It should be mentioned that the total helicity is approximately preserved even if the magnetic field is not frozen into the media but undergoes a small-scale turbulence [Tay]. In this case the fast reconnections of the field trajectories drastically change the local topological characteristics of the field.

However, averaged over the entire domain, the helicity persists for large time intervals.

This phenomenon is based on the fact that small-scale components of the field (the components with wave vectors of large length k) contribute to the total helicity the amount of order $(amplitude)^2/k$, while their contribution to the energy is of order $(amplitude)^2$. Hence, a change of the higher harmonics of the field affects the helicity approximately k times more weakly than it affects the energy.

Analytically, an evolution of the magnetic field $\mathbf{B}$ (div $\mathbf{B} = 0$) in the presence of diffusion is described by the equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\{v, \mathbf{B}\} + \eta \Delta \mathbf{B}.$$

The helicity dissipation over a fixed time δt is

$$\delta \mathcal{H} = 2 \int_M (\text{curl}^{-1}(\eta \Delta \mathbf{B}), \mathbf{B})\mu = -2\eta \int_M (\mathbf{j}, \mathbf{B})\mu,$$

whereas the energy $E = \int_M (\mathbf{B}, \mathbf{B})\mu$ dissipates as

$$\delta E = 2 \int_M (\eta \Delta \mathbf{B}, \mathbf{B})\mu = 2\eta \int_M (\mathbf{j}, \mathbf{j})\mu$$

(here $\mathbf{j} = \text{curl}\,\mathbf{B}$ is the current density). The Schwarz inequality gives the upper bound for $\delta \mathcal{H}$ of order $\eta^{1/2}$: $|\delta \mathcal{H}| \leq |\eta(\delta E)E|^{1/2}$.

The combinatorial arguments of [FrB] show that there are "reconnection pathways" that remove other invariants while changing the helicity only at a rate η^2. Neither of the linking invariants of higher order (≥ 3) defined above for tubes of closed trajectories persist under the reconnection deformations [MSa, FrB].

The reconnection of magnetic lines under magnetic diffusion is similar to the vortex reconnection in a viscid incompressible fluid. We refer to [KiT] for a survey on vortex reconnection and to [Ryl] for other topological properties of various vortex flows.

7.D. Calugareanu invariant and self-linking number

Let a narrow tube around a curve γ in $\mathbb{R}^3$ be filled by the trajectories of a vector field ξ. Suppose that all the ξ-trajectories in the tube are closed and that one of them is the curve γ itself.

The helicity of the field inside the tube is proportional to the linking number lk of any two trajectories inside the pencil:

$$\mathcal{H}(\xi) = lk \cdot Q^2,$$

where Q is the flux of ξ across any section of the tube. A straightforward application of the helicity formulas (4.1–4.2) for a field filling an arbitrary volume, this formula can also be visualized by presenting the tube as consisting of many slim solitori and by counting their mutual helicity (see formula (2.1)).

On the other hand, the linking number lk between the curve γ and a neighboring curve γ' is a quantity assigned to a ribbon bounded by γ and γ'. Precisely, the

linking number is the sum

$$lk = Wr + Tw$$

of the writhing number Wr and the total twisting number Tw defined as follows.

Definitions 7.20. The *writhing number* is the algebraic number of crossovers of the curve $\gamma \subset \mathbb{R}^3$ averaged over all the projection directions:

$$Wr = -\frac{1}{4\pi} \int_{S^1} \int_{S^1} \frac{(\dot{\gamma}(t_1), \dot{\gamma}(t_2), \gamma(t_1) - \gamma(t_2))}{\|\gamma(t_1) - \gamma(t_2)\|^3} \, dt_1 dt_2,$$

where the curve $\gamma = \gamma(t)$ is parametrized by $t \in S^1$ (see, e.g., [Ful]). Just as it is for the average self-crossing number $c(\gamma, \gamma)$ (see Theorem 6.4), the integral above is bounded. Its value is not supposed to be an integer, and it is not a topological invariant. For instance, for a plane (or spherical) curve the writhing number is zero.

The *twist number* is not defined for a curve, but it can be defined for a ribbon. It specifies the total rotation number of the edge $\tilde{\gamma}$ revolving about the "axis" curve γ:

$$Tw = \frac{1}{2\pi} \int_{S^1} \left(\frac{dn(t)}{dt}, n(t), \gamma(t) \right) dt,$$

where $\gamma(t)$ is an arc-parametrization of the curve γ, and the family $n(t)$ consists of the unit normals attached along γ and pointing in the direction of $\tilde{\gamma}$.
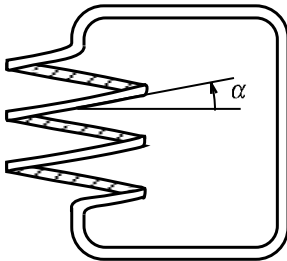
FIGURE 44.   The formula $lk = Wr + Tw$ for a helical ribbon (see [Ful]). Here $lk = n$, $Tw = n \sin \alpha$, $Wr = n(1 - \sin \alpha)$, where α is the pitch angle of a helix, and n is the number of turns.

The formula $lk = Wr + Tw$ is illustrated in Fig. 44. This relation, due to Calugareanu [Cal], was extensively studied along with its numerous applications (e.g., the helical DNA structure) by Fuller [Ful], Pohl [Poh], White [Wh], and in the hydrodynamical context by Berger and Field [B-F], and Moffatt and Ricca [MoR, RiM]. We refer to [MoR] for a derivation of the Calugareanu invariant from basic hydrodynamical principles, as well as for the invariant history and extensive

bibliography. The decomposition $lk = Wr + Tw$ corresponds to the writhe and twist contributions to the helicity of a bundle of field lines, which is a substitution for a ribbon in the hydrodynamical setting.

We also refer to the paper by Bott and Taubes [B-T] for a purely topological notion of the self-linking number of a knot, which has been conceived in the context of the Chern–Simons topological quantum field theory and then decoupled from the group structure involved (see the references therein for the earlier papers by D. Bar-Natan, by A. Guadaguini, M. Martinelli, and M. Mintchev, and by M. Kontsevich). In the next section we describe the relation of the linking numbers to the Chern–Simons functional.

7.E. *Holomorphic linking number*

Many real notions in mathematics have their complex counterparts. The analogies can be as "straightforward" as the correspondence of real and complex manifolds, or of the groups of orthogonal and unitary matrices ($O(n)$, $U(n)$), or much more elaborate, say, the Stiefel–Whitney and Chern characteristic classes of vector bundles. Another nontrivial example is the duality of the homotopy groups π_0 (in the real setting) and π_1 (in the complex setting). It can be understood as follows: The number of connected components (π_0) is a measure of complexity of the complement to a hypersurface in a real manifold. On the other hand, a complex hypersurface does not split a complex manifold, and it can be bypassed. The fundamental group (π_1) measures the complexity of the complement in the latter case. We refer to [Arn23, Kh2] for other examples of informal complexification.

Here we discuss a complex counterpart of the notion of linking number (following the ideas of [At]; see [KhR, FKT, Ger, F-K]). Instead of linking two smooth closed curves in a simply connected real three-manifold, we will deal with an invariant associated to a pair of closed complex curves (Riemann surfaces) in a complex three-dimensional (i.e., of real dimension 6) manifold. In the sketch below we always assume that the described manifolds and forms exist, and we briefly mention the necessary existence conditions.

Remark 7.21. The classical linking number lk is an integer topological invariant equal to the algebraic number of crossings of one curve in $\mathbb{R}^3$ with a two-dimensional surface bounded by the other curve (Fig. 25). The topological invariance of lk and its independence of the choice of surface follow from the fact that the algebraic number of intersections of a closed curve and a closed surface is equal to zero.

The latter invariance can also be viewed as the Stokes formula for δ-type forms supported on closed curves and surfaces (cf. Remark 4.7). The Stokes formula, and more generally, the De Rham theory of smooth differential forms, has a genuine real flavor: One considers real manifolds with boundary and an appropriate orientation, the $\mathbb{Z}/2\mathbb{Z}$-valued invariant.

One argues in [F-K, Kh2] that the Leray theory of meromorphic forms on complex manifolds is an informal complexification of De Rham theory. The Leray residue formula is a higher-dimensional generalization of the Cauchy formula, which gives the value of a contour integral of a meromorphic 1-form via the form's residue at the pole. It "replaces" the Stokes formula in the complexification. Instead of restricting a form to the boundary, one takes the residue of a meromorphic form at the polar set.

To define the Leray residue, let ω be a closed meromorphic k-form on a compact complex n-dimensional manifold M with poles on a nonsingular complex hypersurface $N \subset M$. All poles here and below are supposed to be of the first order. Let ψ be a function defining N in a neighborhood of some point $p \in N$. Then locally, in a certain neighborhood $U(p)$, the k-form ω can be decomposed into the sum

$$(7.3) \qquad\qquad \omega = \frac{d\psi}{\psi} \wedge \alpha + \beta,$$

where α and β are holomorphic in $U(p)$. One can show that the restriction $\alpha\big|_N$ is a well-defined (i.e., independent of ψ) holomorphic $(k-1)$-form (see [Ler]).

Definition 7.22. The *form-residue* res ω of the closed meromorphic k-form ω is the holomorphic $(k-1)$-form on N such that in any neighborhood $U(p)$ of an arbitrary point $p \in N$, it coincides with the form $\alpha\big|_N$ of the decomposition (7.3):

$$\operatorname{res} \omega = \alpha\big|_N.$$

Similarly, one defines the residue in the case of polar sets consisting of several complex hypersurfaces in a general position in M.

Remark 7.23. For a complex manifold M with $h^{n,1}(M) := \dim H^1(M, \Omega^n) = 0$, every holomorphic $(n-1)$-form on N is the residue of some meromorphic n-form on M with poles on N of the first order; see, e.g., [Chr]. This meromorphic n-form on M is defined by its residue on N uniquely up to a holomorphic n-form on M. Note that the condition $h^{n,1}(M) = 0$ in the complex setting can be thought of as an analogue of simple-connectedness of a real manifold.

Now let $C_1, C_2 \subset M$ be two complex closed nonintersecting curves in a complex closed three-fold M: $C_1 \cap C_2 = \emptyset$. Fix some holomorphic differentials α_1 and α_2 on the curves C_1 and C_2, respectively, and a meromorphic 3-form η on M satisfying the following condition: The zero locus of η intersects neither of the curves C_1 and C_2 (e.g., if M is a Calabi–Yau manifold, it possesses a nonvanishing holomorphic 3-form η, unique up to a factor). The number we are going to assign to this pair of curves depends linearly on α_1, α_2, and η^{-1}.

Suppose that there exists a complex surface S_1 in M that contains the complex curve C_1. Denote by β_1 any meromorphic 2-form on S_1 with a polar set on the curve C_1, and such that the residue of this 2-form β_1 is equal to α_1: res $\beta_1|_{C_1} = \alpha_1$.

By virtue of the remark above, such a 2-form β_1 exists as soon as there is a complex surface $S_1 \subset M$ containing the curve C_1 and such that $H^1(S_1, \Omega^2) = 0$.

Definition 7.24 [KhR, FKT]. The *holomorphic linking number* $lk_{\mathbb{C}}$ of the pair of complex curves C_j with chosen holomorphic differentials α_j on them (in the manifold M with the meromorphic form η) is the following sum over all intersection points of the surface S_1 and the curve C_2:

$$(7.4) \qquad lk_{\mathbb{C}}((C_1, \alpha_1), (C_2, \alpha_2)) := \sum_{S_1 \cap C_2} \frac{\beta_1 \wedge \alpha_2}{\eta}.$$

Note that the 3-form $\beta_1 \wedge \alpha_2$ is well-defined at the points of intersection $S_1 \cap C_2$, and the ratio on the right-hand side measures its proportionality coefficient with the 3-form η at the same points.

Unlike the real case, the holomorphic linking number is not integer valued, and it is not an isotopy invariant. Its value can be any complex number, and it depends on the mutual location of the complex curves C_1 and C_2 in M, as well as on the differential forms α_1, α_2, and η involved. However, it will be the same for all additional choices.

Proposition 7.25.

  (*i*) *The holomorphic linking number $lk_{\mathbb{C}}$ is well-defined; i.e., it does not depend on the choice of complex surface $S_1 \supset C_1$ or the meromorphic two-form β_1 on it, provided that res $\beta_1|_{C_1} = \alpha_1$ (Fig. 45).*
  (*ii*) *The value $lk_{\mathbb{C}}$ is a symmetric function of its arguments: One gets the same linking number by embedding the curve C_2 into a complex surface S_2, taking a meromorphic form β_2 such that res $\beta_2|_{C_2} = \alpha_2$, and forming the sum*

$$lk_{\mathbb{C}}((C_1, \alpha_1), (C_2, \alpha_2)) = \sum_{C_1 \cap S_2} \frac{\alpha_1 \wedge \beta_2}{\eta} = lk_{\mathbb{C}}((C_2, \alpha_2), (C_1, \alpha_1)).$$

Proof. Assume that the complex curve C_1 is a transversal intersection of two complex surfaces S_1 and S_1', and each of the surfaces is equipped with a meromorphic 2-form (respectively, β_1 and β_1') whose residues on C_1 are α_1. Define a meromorphic 3-form γ_1 on M with poles (of the first order) on S_1 and S_1' and residues β_1 and $-\beta_1'$, respectively. These conditions on the form γ_1 are consistent.

Indeed, on the intersection of two surfaces the form (second) residue depends on the order in which the repeated residue is taken: It differs by the sign. For example, according to the order, the form $dx \wedge dy/(xy)$ has the second residue 1 or -1 at the origin:

$$\operatorname{res}|_{y=0} \operatorname{res}|_{x=0} \frac{dx \wedge dy}{xy} = \operatorname{res}|_{y=0} \frac{dy}{y} = 1,$$
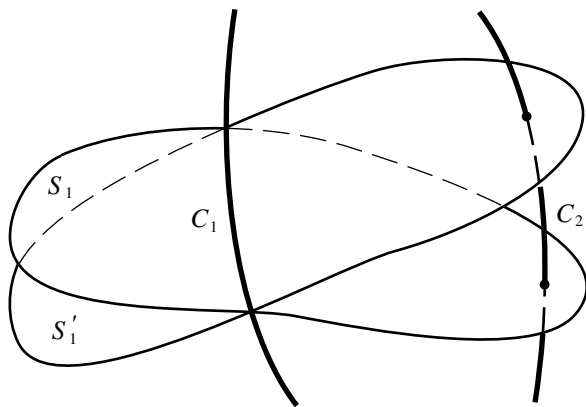
FIGURE 45. The holomorphic linking number of complex curves C_1 and C_2 counts the contributions of the intersections of the curve C_2 with a surface $S_1 \supset C_1$, or equivalently with another surface $S'_1 \supset C_1$.

while

$$\mathrm{res}\,|_{x=0}\,\mathrm{res}\,|_{y=0}\,\frac{dx \wedge dy}{xy} = -\,\mathrm{res}\,|_{x=0}\,\frac{dx}{x} = -1.$$

Similarly, the second residue of the 3-form γ_1 on the curve $C_1 = S_1 \cap S'_1$ is the 1-form α_1 or $-\alpha_1$. For instance,

$$\mathrm{res}\,|_{C_1}\,\mathrm{res}\,|_{S_1}\gamma_1 = \mathrm{res}\,|_{C_1}\beta_1 = \alpha_1.$$

Then, by the definition of the holomorphic linking number (7.4),

$$lk_{\mathbb{C}}\,((C_1, \alpha_1), (C_2, \alpha_2)) = \sum_{S_1 \cap C_2}\frac{(\mathrm{res}\,\gamma_1) \wedge \alpha_2}{\eta},$$

since $\mathrm{res}\,\gamma_1|_{S_1} = \beta_1$. The latter ratio at every point of $S_1 \cap C_2$ is equal to

$$\mathrm{res}\left(\frac{\gamma_1}{\eta} \wedge \alpha_2\right),$$

where $\frac{\gamma_1}{\eta}$ is a meromorphic function on M, and $\frac{\gamma_1}{\eta} \wedge \alpha_2$ is a meromorphic 1-form defined on C_2. Indeed, one can easily see that the equality

$$\frac{(\mathrm{res}\,\gamma_1) \wedge \alpha_2}{\eta} = \mathrm{res}\left(\frac{\gamma_1}{\eta} \wedge \alpha_2\right)$$

holds at every point of the intersection $S_1 \cap C_2$ by doing calculations in local coordinates.

Then $lk_{\mathbb{C}}$ is the sum of residues of the meromorphic 1-form $\frac{\gamma_1}{\eta} \wedge \alpha_2$ on the complex curve C_2 at the poles $S_1 \cap C_2$. By using the surface S'_1 instead of S_1 for the same calculation, one obtains minus the sum of residues of the same 1-form $\frac{\gamma_1}{\eta} \wedge \alpha_2$ on C_2, where the residues are taken at the poles $S'_1 \cap C_2$. The latter follows from the assumption that $\mathrm{res}\,\gamma_1|_{S'_1} = -\beta'_1$.

The Cauchy theorem states that the sum of residues of a meromorphic 1-form on a complex curve is equal to zero. We apply it to the meromorphic 1-form $\frac{\gamma_1}{\eta} \wedge \alpha_2$ on the complex curve C_2. Then the sum of the form's residues at all poles, i.e., at the points of intersection of C_2 with both S_1 and S_1', is equal to zero. This shows that $lk_{\mathbb{C}}$ does not depend on whether we use the surface S_1 or S_1' (statement (i)).

The symmetry of $lk_{\mathbb{C}}$ can be immediately seen if we present C_2 as a transversal intersection of two surfaces S_2 and S_2' and associate to it a meromorphic 3-form γ_2 in the same way as above. Then

$$lk_{\mathbb{C}}\left((C_1, \alpha_1), (C_2, \alpha_2)\right) = \sum_{S_1 \cap S_2' \cap S_2} \operatorname{res}^3 \left(\frac{\gamma_1 \wedge \gamma_2}{\eta}\right),$$

where res^3 is the residue of the meromorphic 3-form

$$\frac{\gamma_1 \wedge \gamma_2}{\eta}$$

at the triple intersections $S_1 \cap S_2' \cap S_2$. The skew symmetry of the wedge product and the sign change when passing to the intersections $S_1 \cap S_1' \cap S_2$ complete the proof of (ii). $\square$

Remark 7.26. The main reason for introducing the complex linking number is that it arises as the "first approximation" of the complex analogue of the Chern–Simons functional (see [FKT, FKR] and Remark 8.9). The standard linking number governs the asymptotics of the classical Chern–Simons functional ([Pol, Wit2], Section 8 below).

Remark 7.27. In a real three-dimensional manifold M, a knot (or link) invariant is a locally constant function on the space of embeddings of a circle (respectively, a union of circles) into the manifold M. In [VasV], V. Vassiliev defined the jump of an invariant as the function assigned to the immersions of the circle with one point of self-intersection and whose value is equal to the difference of the knot invariant on the embeddings "before" and "after" the self-intersection. (Here the notions of "before" and "after" are determined by the orientation of the circle and of the ambient manifold M.) One can iterate the jumps and define the function on immersions with any finite number of self-intersection points.

By definition, the Vassiliev invariant of order k is a knot or link invariant whose jump function vanishes on all immersions with at least $k + 1$ self-intersection points. In particular, one has the following

Proposition 7.28. *The linking number of two curves in $\mathbb{R}^3$ is an invariant of order 1.*

Remark 7.29. The holomorphic linking number $lk_{\mathbb{C}}$ is not defined if the two complex curves C_1 and C_2 intersect, and it tends to infinity as the curves approach each other. We suggest the following "meromorphic" counterpart of the Vassiliev theory.

Let M be a complex three-dimensional manifold equipped with a nonvanishing holomorphic form η. Denote by $\mathcal{M}_j$ the moduli space of all embedded holomorphic curves of fixed genus g_j ($j = 1, 2$) in the complex manifold M. The space $\mathcal{M}_j$ is a finite-dimensional complex manifold by itself (and we assume that its dimension is nonzero). The product $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ can be thought of as a complex analogue of the space of (real) knots or links.

Similar to the real case, it is natural to call the *discriminant* $\Delta \subset \mathcal{M}$ the subset of all configurations in the moduli space $\mathcal{M}$ such that the curves C_1 and C_2 hit each other. The discriminant Δ is a (singular) complex hypersurface in $\mathcal{M}$, and its regular points Δ_0 correspond to simple intersections of the curves C_1 and C_2. Further degenerations of the discriminant variety $\Delta \supset \Delta_0 \supset \Delta_1 \supset \cdots$ are stratified by the number and multiplicity of the intersections.

It would be interesting to define the holomorphic linking number $lk_{\mathbb{C}}$ as a closed differential form on the moduli space $\mathcal{M}$ or on some bundle over it. Since $lk_{\mathbb{C}}$ tends to infinity as the two curves get close to each other, this differential linking form is supposed to have a pole of first order along (the regular part Δ_0 of) the discriminant Δ. In particular, the corresponding residue might be well-defined along Δ_0.

More generally, one can call a *complex link invariant* of complex curves of genera $g_1, g_2, \ldots, g_m$ in a complex three-manifold M any closed meromorphic k-form on the appropriate moduli space $\mathcal{M} := \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_m$ of the holomorphic embeddings in M.

Definition 7.30. A *complex link invariant of order k* is a closed meromorphic form on the moduli space $\mathcal{M}$ whose $(k + 1)$st residue vanishes on all strata Δ_k of the discriminant $\Delta \subset \mathcal{M}$ that correspond to embeddings of complex curves with k points of pairwise intersections.

Problems 7.31. (A) Show that the complex linking form $lk_{\mathbb{C}}$ can be defined as a complex link invariant of order 1. Similarly, one can try to define the complex analogues of Massey products and of other cohomological operations on knots and links.

(B) Give an ergodic interpretation of the holomorphic version of the linking number in the spirit of Section 4.

§8. Asymptotic holonomy and applications

8.A. *Jones–Witten invariants for vector fields*

There is a diversity of subtle invariants for knots and links. For instance, one might consider the knot polynomials (of Alexander, Kauffman, Jones, HOMFLY, Reshetikhin and Turaev, etc.) or the Vassiliev invariants of finite order (see, e.g., [Tu2, VasV]). It is of great interest to extend the domain of such invariants to the case of (divergence-free) vector fields, to "diffuse knots" in the three-space $\mathbb{R}^3$.

From this standpoint, a regular knot is understood as a vector field supported on a single closed curve.

The classical (combinatorial) approach to introducing the knot invariants is based on some type of recurrence relation: One starts with an unknot and defines a precise recipe for how the invariant changes under elementary surgeries (for example, the connected sum). This strategy seemed to be nonapplicable to extending the definitions to vector fields.

The situation changed after Witten's generalization [Wit2] of the Jones polynomial to arbitrary closed 3-manifolds in terms of the asymptotics of the Chern–Simons functional on the space of connections over the manifold. The structure group of the connection gives one more parameter to the problem, and the actual Jones polynomial corresponds to the $SU(2)$-connections.

The extension of Witten's approach from links to "diffuse knots" was started by Verjovsky and Freyer in [V-F], and we present below the main steps of that paper. In the abelian case of the $U(1)$- (or $GL(1)$-) connections the asymptotics in question are essentially determined by the helicity invariant of the corresponding divergence-free vector field. The $GL(n)$-version of the asymptotic monodromy along a nonclosed trajectory of a vector field is provided by Oseledets's multiplicative ergodic theorem [Ose1]. However, the extension of the invariants to the nonabelian case encounters serious obstacles arising from the lack of a nonabelian version of the Birkhoff ergodic theorem on the equality of time and space averages.

Let M be a closed compact real three-manifold M and let $L \subset M$ be a link (a disjoint union $L = \cup_{i=1}^{n} C_i$ of smoothly embedded circles C_i). Further, let $P = M \times G$ be the G-principal bundle over M, where the structure group G might be $U(1)$ or $SU(2)$.

Denote by $\mathcal{A}$ the space of all connections in the (trivial) bundle P. It can be identified with the affine space $\Lambda^1(M, \mathfrak{g})$ of 1-forms on M with values in the Lie algebra $\mathfrak{g}$ of G. Finally, let $\tilde{G} = C^\infty(M, G)$ be the current group of fiber-preserving automorphisms of P.

Definition 8.1. The *Jones–Witten invariant* of a link $L \subset M$ is the following function of k:

$$W_L(k) = \int_{\mathcal{A}/\tilde{G}} \left\{ \exp(ik \int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)) \right.$$

$$\left. \cdot \prod_{C_i \subset L} \mathrm{tr}(P \exp \int_{C_i} A) \right\} \cdot DA,$$

where $P \exp$ is the path-ordered exponential integral, and DA is "an appropriate measure on the moduli space of the connections." From the mathematical point of view, neither DA nor W has a sound definition.

Witten showed in [Wit2] that for $M = S^3$ and $G = SU(2)$ this corresponds to the Jones polynomial (in k) for the link L. Though justification of the meaning of this integral is still not complete, it looks a lot simpler for an abelian group G, say $U(1)$:

$$(8.1) \qquad W_L(k) = \int_{\mathcal{A}/\tilde{G}} \left\{ \exp(ik \int_M A \wedge dA) \cdot \prod_{C_i \subset L} (\exp \int_{C_i} A) \right\} \cdot DA.$$

One can think of $\int_M A \wedge dA$ as a quadratic form $Q(A)$ on $\mathcal{A}$, while the line integral $\int_{C_i} A$ is regarded as a linear functional (the so-called *De Rham current*) $I_{C_i}(A)$ evaluated at the 1-form A.

For the abelian case (see [SchA, Pol]), the path integral modulo factors related to a regularization and topology of the manifold M is equal to

$$(8.2)$$
$$W_L(k) = \text{const} \cdot \exp \left\{ \frac{i}{2k} \sum_{i,j} \langle I_{C_i}, d^{-1} I_{C_j} \rangle \right\} = \text{const} \cdot \exp \left\{ \frac{i}{2k} \sum_{i,j} lk(C_i, C_j) \right\}.$$

The regularization is needed to define the linking number for each curve C_i with itself (cf. the definition of self-linking number in Section 7.D). The topological factor, being the value of $W_L(k)$ in the case without any link ($L = \emptyset$), is the Ray–Singer torsion of the manifold M [SchA].

Remark 8.2. Heuristically, one computes here a quadratic Gaussian integral of the type

$$\int_{\mathbb{R}^n} e^{ik\langle x, Qx \rangle} e^{i\langle b, x \rangle} (\pi^{-n/2}) \, dx,$$

which, upon the extraction of a complete square, is equal to

$$e^{\frac{i}{2k} \langle b, Q^{-1}b \rangle} \int_{\mathbb{R}^n} e^{ik\langle x+Q^{-1}b, Q(x+Q^{-1}b) \rangle} (\pi^{-n/2}) \, dx$$
$$= e^{\frac{i}{2k} \langle b, Q^{-1}b \rangle} (\det Q)^{-1/2} (e^{(\frac{i\pi}{4} \cdot \text{sign } Q)}).$$

One can apply this formula to the (completion of the) infinite-dimensional space $\mathcal{A} = \Omega^1(M, \mathfrak{g})$ in the case of the quadratic form $Q(A) = \int_M A \wedge dA$. Since the form Q is degenerate, the integration is carried out only along a subspace in the space $\mathcal{A}$ transversal to the kernel of Q. This corresponds to integration over the $\tilde{G}$-quotient of the space $\mathcal{A}$; see [Wit2, V-F]. Although this differs from the above case of a nondegenerate form, here we are interested only in the factor $e^{\frac{i}{2k} \langle b, Q^{-1}b \rangle}$, which has a straightforward analogue.

In our context, this factor turns out to be the linking term:

$$e^{\frac{i}{2k} \langle b, Q^{-1}b \rangle} = \exp \left\{ \frac{i}{2k} \sum_{i,j} \langle I_{C_i}, d^{-1} I_{C_j} \rangle \right\}.$$

The ergodic ("diffuse") version of this approach has to do with notions of asymptotic and average holonomy. (One can think of diffusing the knot as the way of its regularization: The neighboring trajectories can be regarded as a framing. In particular, it allows one to determine the knot self-linking as the linking number of the knot with its shift in the direction of the frame.)

Definition 8.3. The *asymptotic holonomy* of a connection A along the trajectory $\Gamma_\xi(p) = \{g_\xi^t \mid t \geq 0\}$ of a vector field ξ issuing from a point $p \in M$ is the following element of the Lie group G:

$$P \exp \int_{\Gamma_\xi(p)} A := \lim_{T \to \infty} P \exp \int_{\{g_\xi^t \mid 0 \leq t \leq T\}} \left(\frac{1}{T} A \right).$$

The last integral is defined by the limiting procedure $T \to \infty$, due to the trivialization of the bundle $P = M \times G$ (or by means of a system of short paths used for the asymptotic linking number; see Definition 4.13).

It can also be thought of as follows. The (indefinite) integration

$$P \exp \int_{\{g_\xi^t \mid t \geq 0\}} A$$

along the trajectory $\Gamma_\xi(p)$ defines a curve in the Lie group G. Choose the one-parameter subgroup in G approximating this curve as $t \to \infty$. Then the asymptotic holonomy is the point $t = 1$ on the subgroup. The existence of this limit for an arbitrary group G is obscure. However, in some cases, the limiting eigenvalues for almost all initial points $p \in M$ are provided by the multiplicative ergodic theorem [Ose1].

Though in the nonabelian case no simple answer for the space average of the asymptotic holonomy is known (there is no matrix analogue to the Birkhoff ergodic theorem on the equality of time- and space-averages), we present the would-be definition in "full" generality (see [V-F]).

Definition 8.4. The *average holonomy* $\mathrm{hol}_{\xi,\mu}(A)$ of a connection A on a divergence-free field ξ preserving the measure μ on M is the group exponent of the Lie algebra element $\int_M A(\xi)\mu$.

Remark 8.5. In general, neither the average holonomy $\mathrm{hol}_{\xi,\mu}(A)$ nor its conjugacy class in the group G is gauge invariant (i.e., preserved under the change of the connection A to $A + \epsilon([A, f] + df)$ for an arbitrary $f \in C^\infty(M, \mathfrak{g})$).

In the case of abelian (say, $GL(1)$) connection, the form A is a real-valued 1-form on M, and the ordered exponent $P \exp$ becomes an ordinary exponent. Then the holonomy $\mathrm{hol}_{\xi,\mu}(A)$ is gauge invariant, and the above definitions exactly correspond to the ergodic interpretation of the Hopf (helicity) functional in terms of average linking number considered above:

Theorem 8.5 (= Helicity Theorem 4.4). *For an abelian group G the multiplicative average of the asymptotic holonomy over the entire manifold M coincides with the average holonomy calculated by total integration of the "infinitesimal transforms" $A(\xi)$:*

$$\exp \int_M \left(\int_{\Gamma_\xi(p)} A \right) \mu_p = \text{hol}_{\xi,\mu}(A).$$

The latter identity of the two invariants suggests the following definition.

Definition 8.6 [V-F]**.** The *Jones–Witten functional* for a divergence free *vector field ξ* on a closed three-manifold M endowed with a measure μ is the expression

$$W_{\xi,\mu}(k) = \int_{A/\tilde{G}} \{\exp(ik \int_M \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)) \cdot \text{tr}(\text{hol}_{\xi,\mu}(A))\} \cdot DA,$$

where the average holonomy $\text{hol}_{\xi,\mu}(A)$ is defined above.

Remark 8.7. Note that the case of an actual knot or link $L = \bigcup_{i=1}^n C_i$ can be understood as a particular case of this definition for a "δ-type" measure μ supported on a finite number of curves $\{C_i\}$.

Assume now that M is a closed three-manifold, μ is a smooth volume form on M, and ξ is a null-homologous trivial vector field on M; i.e., the two-form $i_\xi \mu$ is exact: $i_\xi \mu = d\theta$ for some 1-form θ. The case of the abelian connection can be handled completely:

Theorem 8.8 [V-F]**.** *For a topologically trivial linear bundle over M (with $G = U(1)$ or $GL(1)$), the Jones–Witten functional for the vector field ξ reduces to its helicity invariant:*

$$W_{\xi,\mu}(k) = \text{const} \cdot \exp(\frac{i}{2k} \int_M d\theta \wedge \theta).$$

Proof sketch. The average holonomy in the abelian case is

$$\text{hol}_{\xi,\mu}(A) = \exp \int_M A(\xi) \wedge \mu = \exp \int_M A \wedge d\theta = \exp(I_\xi(A)),$$

where I_ξ is the De Rham current corresponding to the field ξ. (The "diffuse" term

$$\text{hol}_{\xi,\mu}(A) = \exp \int_M A(\xi) \wedge \mu = \exp \int_M \int_{\Gamma_\xi(p)} A \wedge \mu_p$$

replaces the "discrete" counterpart

$$\prod_{C_i \subset L} (\exp \int_{C_i} A) = \exp(\sum_{C_i \subset L} \int_{C_i} A)$$

in (8.1).) Then the expression (8.2) for the abelian case becomes

$$W_{\xi,\mu}(k) = \text{const} \cdot \exp\{\frac{i}{2k}\langle I_\xi, d^{-1}I_\xi\rangle\} = \text{const} \cdot \exp(\frac{i}{2k}\int_M d\theta \wedge \theta).$$

$$\square$$

Remarks 8.9. The *Chern–Simons functional*

$$\text{CS}(A) = \int_M \text{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)$$

on G-connections $\{A\}$ over real three-dimensional manifolds M has a *complex analogue* for Calabi–Yau manifolds, or, more generally, for any three-dimensional complex manifold N; see [Wit3]:

$$\text{CS}_{\mathbb{C}}(A) = \int_N \text{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \wedge \eta,$$

where η is a holomorphic (or meromorphic) 3-form on N. In the case of the abelian group $G = GL(1, \mathbb{C})$ and a complex link L, being a disjoint union of *complex curves* C_i with holomorphic differentials α_i on them, the asymptotics of the corresponding complex analogue of the Jones–Witten functional W_L is given by the holomorphic linking number $lk_{\mathbb{C}}((C_i, \alpha_i), (C_j, \alpha_j))$ defined in Section 7.E (see [FKT, FKR, Ger]).

Remarks 8.10. The higher linking numbers introduced in Section 7.B arise in the calculation of correlators in Chern–Simons theories in dimensions greater than 3 (see [FNRS]).

A higher-dimensional version of the Chern–Simons path integral can be regarded as a nonabelian counterpart of the corresponding hydrodynamical integral. Being an example of so-called topological field theories, by its very definition it does not require a metric to specify the action functional. Hence, all *gauge-invariant* observables in the theory are topologically invariant, provided that the measure in the path integral does not spoil the invariance under diffeomorphisms.

Let $\{A\}$ be the space of $U(1)$-connections on a manifold M^{2m+1}; DA is a shift-invariant integration measure. For a collection of cycles $C_1, \ldots, C_r$ of dimensions $\dim C_i = 2d_i + 1, i = 1, \ldots, r$, define the gauge-invariant functional

$$\Phi_{\{C_1,\ldots,C_r\}}(A) := \prod_{i=1}^{r} \exp\left(\int_{C_i} A \wedge (dA)^{d_i}\right).$$

Suppose that the cycles obey the linking condition (7.1): $\sum_{i=1}^{r}(m - d_i) = m + 1$. Then asymptotically for large k the expectation value of the functional Φ, that is,

$$< \Phi_{\{C_1,\ldots,C_r\}}(A) >= \int \Phi_{\{C_1,\ldots,C_r\}}(A) \cdot \exp\left(\frac{ik}{2m+1}\int_M A \wedge (dA)^m\right) DA,$$

is given by the exponent of the mutual linking number for the collection of cycles: $\exp(lk(C_1, \ldots, C_r)/k^{r-1})$, where the number $lk(C_1, \ldots, C_r)$ is the linking number of, say, C_r with the intersection of all other cycles; see Section 7.B. (To avoid the contribution of the self-linking of the cycles into the integral, one assumes the so-called *normal ordering* of the operators involved.) If the linking condition is not fulfilled, but there are sublinks saturating the condition, then the leading term in the asymptotics is given by the mutual linking numbers of these sublinks.

Remarks 8.11. The above holonomy functional can be regarded as a counterpart of the Radon transform: given a Lie group G it sends a gauge equivalence class of the G-connections on M to a G-valued functional on the space of loops in M.

The value of the *holonomy functional* on a loop Γ is the holonomy of a connection A around Γ. In the abelian case $(G = \mathbb{R})$ the Radon transform associates to a one-form θ on M the corresponding functional I_θ on the free loop space $\mathcal{L}M$ (the space of smooth maps $S^1 \to M$):

$$I_\theta(\Gamma) = \int_\Gamma \theta.$$

In [Bry2], Brylinski characterizes the range of the Radon transform as the set of functionals on $\mathcal{L}M$ obeying a certain system of second-order linear PDE (called the Radon–John system). The necessary and sufficient conditions are constraints on the partial derivatives $\partial^2 I_\theta/\partial x_k^i \partial x_l^j$, where the coordinates $\{x_k^i\}$ are the Fourier components of small variations of the curve Γ. In dimension 2, this system gives rise to the hypergeometric systems in the spirit of [GGZ]. A nonabelian counterpart of the Radon–John equations involves the bracket iterated integrals (see [Bry2]).

Note that in three dimensions the Radon transform displays the kind of functionals on vector fields that can be defined as fluxes of fields through surfaces bounded by embedded curves (or, the same, as the average linking number of the fields and the curves). Indeed, the embedded nonparametrized curves in $\mathbb{R}^3$ form a subset in the dual $S\,\mathrm{Vect}(\mathbb{R}^3)^*$ of the Lie algebra of divergence-free vector fields in the space (see Section VI.3). A curve $\Gamma \subset \mathbb{R}^3$ defines the functional whose value at a divergence-free field ξ is the flux of ξ through Γ.

To relate it to the description above, fix a vector field ξ and assume that μ is a volume form in the space. Let θ be a one-form such that $i_\xi \mu = d\theta$ (ξ is the vorticity field for θ). Then $I_\theta(\Gamma) := \int_\Gamma \theta = \{\text{flux of } \xi \text{ through } \Gamma\}$ can be regarded as the functional on the Γ's. A regular element of the dual space $S\,\mathrm{Vect}(\mathbb{R}^3)^*$ is a "diffuse" loop Γ, a divergence-free vector field η (see Section I.3), while the pairing is

$$I_\theta(\eta) := \int_{\mathbb{R}^3} \theta(\eta)\mu = \mathcal{H}(\xi, \eta).$$

8.B. Interpretation of Godbillon–Vey-type characteristic classes

Let $\mathcal{F}$ be a cooriented foliation of codimension 1 on the oriented closed manifold M, and θ a 1-form determining this foliation. Then $d\theta = \theta \wedge w$ for a certain 1-form w.

Proposition 8.12 (see, e.g., [Fuks]). *The form $w \wedge dw$ is closed, and its cohomology class does not depend on the choices of θ and w.*

Definition 8.13. The cohomology class of the form $w \wedge dw$ in $H^3(M, \mathbb{R})$ is called the *Godbillon–Vey class* of the foliation $\mathcal{F}$.

On a three-dimensional manifold this class is defined by its value on the fundamental 3-cycle:

$$GV(\mathcal{F}) = \int_M w \wedge dw.$$

Let $\mathbf{v}$ be an arbitrary vector field with the sole restriction $\theta(\mathbf{v}) = 1$, and let $L_{\mathbf{v}}^k$ denote the kth Lie derivative along $\mathbf{v}$.

Theorem 8.14 (see [Sul, Th1]). $GV(\mathcal{F}) = -\int_M L_{\mathbf{v}}^2\theta \wedge d\theta$.

If M^3 is a manifold equipped with a volume form μ, the class GV admits an ergodic interpretation in terms of the asymptotic Hopf invariant of a special vector field.

Define the vector field ζ by the relation

$$i_\zeta \mu = L_{\mathbf{v}}^2\theta \wedge \theta.$$

Corollary 8.15 [Tab1]. *The vector field ζ is null-homologous, and its asymptotic Hopf invariant is equal to the Godbillon–Vey invariant of the foliation $\mathcal{F}$.*

Proofs. By the homotopy formula (see Section I.7.B)

$$L_{\mathbf{v}}\theta = di_{\mathbf{v}}\theta + i_{\mathbf{v}}d\theta = i_{\mathbf{v}}\theta \wedge w = w - f\theta,$$

where the function f is $f = w(\mathbf{v})$. This implies that $(L_{\mathbf{v}}\theta) \wedge \theta = w \wedge \theta = -d\theta$, and moreover,

$$(L_{\mathbf{v}}^2\theta) \wedge \theta = L_{\mathbf{v}}((L_{\mathbf{v}}\theta) \wedge \theta) = L_{\mathbf{v}}(-d\theta) = -dL_{\mathbf{v}}\theta = -d(w - f\theta).$$

Hence we can take $w' = w - f\theta$ as a new 1-form w in the definition of the Godbillon–Vey class. Theorem 8.10 readily follows:

$$\int_M L_{\mathbf{v}}^2\theta \wedge d\theta = \int_M L_{\mathbf{v}}^2\theta \wedge \theta \wedge w' = -\int_M dw' \wedge w' = -GV(\mathcal{F}).$$

The null-homologous property for the field ζ also follows from the fact that the 2-form $(L_{\mathbf{v}}^2 \theta) \wedge \theta = i_\zeta \mu$ is a complete differential. Furthermore, the asymptotic Hopf invariant of ζ is

$$\mathcal{H}(\zeta) = \int_M i_\zeta \mu \wedge d^{-1}(i_\zeta \mu) = \int_M dw' \wedge w' = GV(\mathcal{F}).$$

$\square$

Remark 8.16 [Sul, Th1, Tab1]. Having defined an auxiliary vector field ξ (tangent to the leaves of $\mathcal{F}$) by the relation

$$i_\xi \mu = L_{\mathbf{v}} \theta \wedge \theta,$$

one may argue that it measures the rotation of the tangent planes to the foliation in the transversal direction $\mathbf{v}$. Namely, the direction of ξ is the axis of rotation, and the modulus of ξ is the angular velocity of the rotation. Then one may say that ζ measures the acceleration of the rotation, and the above statement reads: $GV(\mathcal{F})$ is the asymptotic Hopf invariant of this rotation acceleration field.

As an element of $H^3(M, \mathbb{R})$, the Godbillon–Vey class on manifolds of higher dimensions is determined by its values on 3-cycles. Any such value coincides with the asymptotic Hopf invariant of the corresponding field ζ, constructed for the induced foliation on the 3-cycle.

Similarly, one can define the asymptotic and integral Bennequin invariants for a null-homologous vector field on a contact simply connected three-manifold (see [Tab1]). These invariants generalize the classical Bennequin definition of the self-linking number of a curve transverse to the contact structure [Ben]. Interesting polynomial invariants of Legendrian curves (and more generally, of framed knots) in a solid torus, generalizing the Bennequin invariant, have been introduced by Aicardi [Aic] (see also [FuT, Fer, Pl2]).

In conclusion, we refer to [DeR, SchL, SchS, GPS, Sul] and references therein for various questions related to structure cycles, asymptotic cycles, approximations of cycles by flows and foliations, and the corresponding smoothness conditions.

Problems 8.17. (A) Give an ergodic interpretation of the global real-valued invariant of three-dimensional CR-manifolds found in [B-E].

Roughly speaking, a CR-structure on a $(2n+1)$-dimensional manifold is defined by choosing an n-dimensional integrable subbundle $T^{1,0}M$ of the complexified tangent bundle of M. In particular, this subbundle determines a distribution of the corresponding contact elements on M. The CR-structure gives rise to a real-valued (Chern–Simons-type) 3-form (defined modulo an exact form) on the manifold.

(B) It would be interesting to consider whether similar techniques can be applied to generalize the Casson invariant and the Floer homology of homological 3-spheres to aspherical $(4k - 1)$-manifolds with an additional structure (say, to contact manifolds); see [CLM, Arn24].

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Chapter IV

Differential Geometry of Diffeomorphism Groups

In 1963 E.N. Lorenz stated that a two-week forecast would be the theoretical bound for predicting the future state of the atmosphere using large-scale numerical models [Lor]. Modern meteorology has currently reached a good correlation of the observed versus predicted for roughly seven days in the northern hemisphere, whereas this period is shortened by half in the southern hemisphere and by two-thirds in the tropics for the same level of correlation [Kri]. These differences are due to a very simple factor: the available data density.

The mathematical reason for the differences and for the overall long-term unreliability of weather forecasts is the exponential scattering of ideal fluid (or atmospheric) flows with close initial conditions, which in turn is related to the negative curvatures of the corresponding groups of diffeomorphisms as Riemannian manifolds. We will see that the configuration space of an ideal incompressible fluid is highly "nonflat" and has very peculiar "interior" and "exterior" differential geometry.

"Interior" (or "intrinsic") characteristics of a Riemannian manifold are those persisting under any isometry of the manifold. For instance, one can bend (i.e., map isometrically) a sheet of paper into a cone or a cylinder but never (without stretching or cutting) into a piece of a sphere. The invariant that distinguishes Riemannian metrics is called Riemannian curvature. The Riemannian curvature of a plane is zero, and the curvature of a sphere of radius R is equal to R^{-2}. If one Riemannian manifold can be isometrically mapped to another, then the Riemannian curvature at corresponding points is the same.

The Riemannian curvature of a manifold has a profound impact on the behavior of geodesics on it. If the Riemannian curvature of a manifold is positive (as for a sphere or for an ellipsoid), then nearby geodesics oscillate about one another in most cases, and if the curvature is negative (as on the surface of a one-sheet hyperboloid), geodesics rapidly diverge from one another.

It turns out that diffeomorphism groups equipped with a one-sided invariant metric look very much like negatively curved manifolds. In Lagrangian mechanics a motion of a natural mechanical system is a geodesic line on a manifold-configuration space in the metric given by the difference of kinetic and potential

energy. In the case at hand the geodesics are motions of an ideal fluid. Therefore, calculation of the curvature of the diffeomorphism group provides a great deal of information on instability of ideal fluid flows.

In this chapter we discuss in detail curvatures and metric properties of the groups of volume-preserving and symplectic diffeomorphisms, and present the applications of the curvature calculations to reliability estimates for weather forecasts.

§1. The Lobachevsky plane and preliminaries in differential geometry

1.A. The Lobachevsky plane of affine transformations

We start with an oversimplified model for a diffeomorphism group: the (two-dimensional) group G of all affine transformations $x \mapsto a + bx$ of a real line (or, more generally, consider the $(n + 1)$-dimensional group G of all dilations and translations of the n-dimensional space $\mathbb{R}^n : x, a \in \mathbb{R}^n, b \in \mathbb{R}_+$).

Regard elements of the group G as pairs (a, b) with positive b or as points of the upper half-plane (half-space, respectively). The composition of affine transformations of the line defines the group multiplication of the corresponding pairs:

$$(a_2, b_2) \circ (a_1, b_1) = (a_2 + a_1 b_2, b_1 b_2).$$

To define a one-sided (say, left) invariant metric on G (and the corresponding Euler equation of the geodesic flow on G; see Chapter I), one needs to specify a quadratic form on the tangent space of G at the identity.

Fix the quadratic form $da^2 + db^2$ on the tangent space to G at the identity $(a, b) = (0, 1)$. Extend it to the tangent spaces at other points of G by left translations.

Proposition 1.1. *The left-invariant metric on G obtained by the procedure above has the form*

$$ds^2 = \frac{da^2 + db^2}{b^2}.$$

Proof. The left shift by an element (a, b) on G has the Jacobian matrix $\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$. Hence, the quadratic form $da^2 + db^2$ on the tangent space $T_{(0,1)}G$ at the identity is the pullback of the quadratic form $(da^2 + db^2)/b^2$ on $T_{(a,b)}G$ at the point (a, b). □

Note that starting with *any* positive definite quadratic form, one obtains an isometric manifold (up to a scalar factor in the metric).

Definition 1.2. The Riemannian manifold G equipped with the metric $ds^2 = (da^2 + db^2)/b^2$ is called the *Lobachevsky plane* Λ^2 (respectively, the *Lobachevsky space* Λ^{n+1} for $x, a \in \mathbb{R}^n$).

Proposition 1.3. *Geodesic lines (i.e., extremals of the length functional) are the vertical half lines ($a = \text{const}, b > 0$) and the semicircles orthogonal to the a-axis (or a-hyperplane, respectively).*

Here vertical half lines can be viewed as semicircles of infinite radius; see Fig. 46.
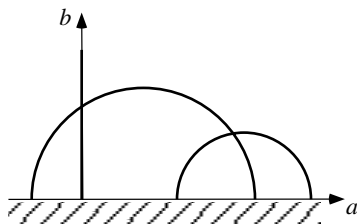
FIGURE 46.   The Lobachevsky plane Λ^2 and geodesics on it.

Proof. Reflection of Λ^2 in a vertical line or in a circle centered at the a-axis is an isometry. □

Note that two geodesics on Λ^2 with close initial conditions diverge exponentially from each other (in the Lobachevsky metric). On a path only few units long, a deviation in initial conditions grows 100 times larger. The reason for practical indeterminacy of geodesics is the negative curvature of the Lobachevsky plane (it is constant and equal to -1 in the metric above). The curvature of the sphere of radius R is equal to R^{-2}. The Lobachevsky plane might be regarded as a sphere of imaginary radius $R = \sqrt{-1}$.

Problem 1.4 (B.Ya. Zeldovich). Prove that medians of every geodesic triangle in the Lobachevsky plane meet at one point. (Hint: Prove it for the sphere; then regard the Lobachevsky plane as the analytic continuation of the sphere to the imaginary values of the radius.)

1.B. Curvature and parallel translation

This section recalls some basic notions of differential geometry necessary in the sequel. For more extended treatment see [Mil3, Arn16, DFN].

Let M be a Riemannian manifold (one can keep in mind the Euclidean space $\mathbb{R}^n$, the sphere S^n, or the Lobachevsky space Λ^n of the previous section as an example); let $x \in M$ be a point of M and $\xi \in T_x M$ a vector tangent to M at x.

Denote by any of $\{\gamma(x, \xi, t)\} = \{\gamma(\xi, t)\} = \{\gamma(t)\} = \gamma$ the *geodesic line* on M, with the initial velocity vector $\xi = \dot{\gamma}(0)$ at the point $x = \gamma(0)$. (Geodesic lines can be defined as extremals of the action functional: $\delta \int \dot{\gamma}^2 dt = 0$. It is called the "principle of least action.")

Parallel translation along a geodesic segment is a special isometry mapping the tangent space at the initial point onto the tangent space at the final point, depending smoothly on the geodesic segment, and obeying the following properties.

(1) Translation along two consecutive segments coincides with translation along the first segment composed with translation along the second.
(2) Parallel translation along a segment of length zero is the identity map.
(3) The unit tangent vector of the geodesic line at the initial point is taken to the unit tangent vector of the geodesic at the final point.

Example 1.5. The usual parallel translation in Euclidean space satisfies properties (1)–(3).

The isometry property, along with the property (3), implies that the angle formed by the transported vector with the geodesic is preserved under translation. This observation alone determines parallel translation in the two-dimensional case, i.e., on surfaces; see Fig. 47.
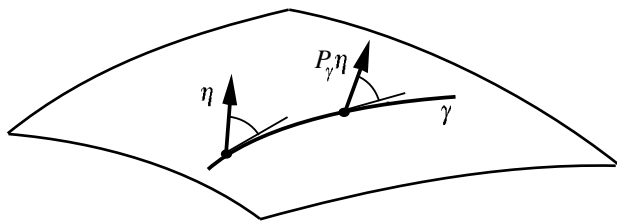

FIGURE 47.   Parallel translation along a geodesic line γ.

In the higher-dimensional case, parallel translation is not determined uniquely by the condition of preserving the angle: One has to specify the plane containing the transported vector.

Definition 1.6. *(Riemannian) parallel translation* along a geodesic is a family of isometries obeying properties (1)–(3) above, and for which the translation of a vector η along a short segment of length t remains tangent (modulo $O(t^2)$-small correction as $t \to 0$) to the following two-dimensional surface. This surface is formed by the geodesics issuing from the initial point of the segment with the velocities spanned by the vector η and by the velocity of the initial geodesic.

Remark 1.7. A physical description of parallel translation on a Riemannian manifold can be given using the adiabatic (slow) transportation of a pendulum along

a path on the manifold (Radon; see [Kl]). The plane of oscillations is parallelly translated.

A similar phenomenon in optics is called the inertia of the polarization plane along a curved ray (see [Ryt, Vld]). It is also related to the additional rotation of a gyroscope transported along a closed path on the surface (or in the oceans) of the Earth, proportional to the swept area [Ish]. A modern version of these relations between adiabatic processes and connections explains, among other things, the Aharonov–Bohm effect in quantum mechanics. This version is also called the Berry phase (see [Berr, Arn23]).

Definition 1.8. The *covariant derivative* $\nabla_\xi \bar{\eta}$ of a tangent vector field $\bar{\eta}$ in a direction $\xi \in T_x M$ is the rate of change of the vector of the field $\bar{\eta}$ that is parallel-transported to the point $x \in M$ along the geodesic line γ having at this point the velocity ξ (the vector observed at x at time t must be transported from the point $\gamma(\xi, t)$).

Note that the vector field $\dot{\gamma}$ along a geodesic line γ obeys the equation $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$.

Remark 1.9. For calculations of parallel translations in the sequel we need the following explicit formulas. Let $\gamma(x, \xi, t)$ be a geodesic line in a manifold M, and let $P_{\gamma(t)} : T_{\gamma(0)} M \to T_{\gamma(t)} M$ be the map that sends any $\eta \in T_{\gamma(0)} M$ to the vector

$$(1.1) \qquad P_{\gamma(\xi,t)}\eta := \frac{1}{t} \frac{d}{d\tau}\bigg|_{\tau=0} \gamma(x, \xi + \eta\tau, t) \in T_{\gamma(t)} M.$$

The mapping $P_{\gamma(t)}$ approximates parallel translation along the curve γ in the following sense. The *covariant derivative* $\nabla_\xi \bar{\eta}$ of the field $\bar{\eta}$ in the direction of the vector $\xi \in T_x M$ is equal to

$$(1.2) \qquad \nabla_\xi \bar{\eta} := \frac{d}{dt}\bigg|_{t=0} P_{\gamma(t)}^{-1} \bar{\eta}(\gamma(\xi, t)) \in T_x M.$$

Remark 1.10. The following properties uniquely determine the covariant derivative on a Riemannian manifold and can be taken as its axiomatic definition (see, e.g., [Mil3, K-N]):

(1) $\nabla_\xi v$ is a bilinear function of the vector ξ and the field v;
(2) $\nabla_\xi f v = (L_\xi f)v + f(\nabla_\xi v)$, where f is a smooth function and $L_\xi f$ is the derivative of f in the direction of the vector ξ in $T_x M$;
(3) $L_\xi \langle v, w \rangle = \langle \nabla_\xi v, w(x) \rangle + \langle v(x), \nabla_\xi w \rangle$; and
(4) $\nabla_{v(x)} w - \nabla_{w(x)} v = \{v, w\}(x)$.

Here $\langle\,,\,\rangle$ is the inner product defined by the Riemannian metric on M, and $\{v, w\}$ is the Poisson bracket of two vector fields v and w. In local coordinates $(x_1, \ldots, x_n)$

on M the Poisson bracket is given by the formula

$$\{v, w\}_j = \sum_{i=1}^n \left(v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_j}{\partial x_i} \right).$$

Parallel translation along any curve is defined as the limit of parallel translations along broken geodesic lines approximating this curve. The increment of a vector after the translation along the boundary of a small region on a smooth surface is (in the first approximation) proportional to the area of the region.

Definition 1.11. The (Riemannian) *curvature tensor* Ω describes the infinitesimal transformation in a tangent space obtained by parallel translation around an infinitely small parallelogram. Given vectors $\xi, \eta \in T_x M$, consider a curvilinear parallelogram on M "spanned" by ξ and η. The main (bilinear in ξ, η) part of the increment of any vector in the tangent space $T_x M$ after parallel translation around this parallelogram is given by a linear operator $\Omega(\xi, \eta) : T_x M \to T_x M$. The action of $\Omega(\xi, \eta)$ on a vector $\zeta \in T_x M$ can be expressed in terms of covariant differentiation as follows:

$$(1.3) \qquad \Omega(\xi, \eta)\zeta = (-\nabla_{\bar{\xi}} \nabla_{\bar{\eta}} \bar{\zeta} + \nabla_{\bar{\eta}} \nabla_{\bar{\xi}} \bar{\zeta} + \nabla_{\{\bar{\xi}, \bar{\eta}\}} \bar{\zeta})|_{x=x_0},$$

where $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ are any vector fields whose values at the point x are ξ, η, and ζ. The value of the right-hand side does not depend on the extensions $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ of the vectors ξ, η, and ζ.

The *sectional curvature* of M in the direction of the 2–plane spanned by two orthogonal unit vectors $\xi, \eta \in T_x M$ in the tangent space to M at the point x is the value

$$(1.4) \qquad C_{\xi\eta} = \langle \Omega(\xi, \eta)\xi, \eta \rangle.$$

For a pair of arbitrary (not necessarily orthonormal) vectors the sectional curvature $C_{\xi\eta}$ is

$$(1.5) \qquad C_{\xi\eta} = \frac{\langle \Omega(\xi, \eta)\xi, \eta \rangle}{\langle \xi, \xi \rangle \langle \eta, \eta \rangle - \langle \xi, \eta \rangle^2}.$$

Example 1.12. The sectional curvature at every point of the Lobachevsky plane (Section 1) is equal to -1. (Hint: use the explicit description of the geodesics in the plane. See also Section 2 for general formulas for sectional curvatures on Lie groups.)

Definitions 1.13. The *normalized Ricci curvature* (at a point x) in the direction of a unit vector ξ is the average of the sectional curvatures of all tangential 2-planes containing ξ. It is equal to $r(\xi)/(n-1)$, where the *Ricci curvature* $r(\xi)$ is the value $r(\xi) = \sum_i C_{\xi e_i} = \sum_i \langle \Omega(\xi, e_i)\xi, e_i \rangle$, calculated in any orthonormal basis $e_1, \ldots, e_n$ of the tangent space $T_x M$.

The *normalized scalar curvature* at a point x is the average of all sectional curvatures at the point. It is equal to $\rho/[n(n-1)]$, the *scalar curvature ρ* being the sum $\rho = r(e_i) + \cdots + r(e_n) = 2 \sum\limits_{i<j} C_{e_i e_j}$ (see, e.g., [Mil4]).

1.C. Behavior of geodesics on curved manifolds

From the definition of curvature one easily deduces the following

Proposition 1.14. *The distance $y(t)$ between two infinitely close geodesics on a surface satisfies the differential equation*

$$(1.6) \qquad \ddot{y} + Cy = 0,$$

where $C = C(t)$ is the Riemannian curvature of the surface along the geodesic.

In order to describe in general how the curvature tensor affects the behavior of geodesics, we look at a variation $\gamma_\alpha(t)$ of a geodesic $\gamma = \gamma_0(t)$. For each α sufficiently close to 0, the curve γ_α is a geodesic whose initial condition is close to that of γ. The field $\xi(t) = \frac{d}{d\alpha}\big|_{\alpha=0}\gamma_\alpha(t)$ (defined along γ) is called the *vector field of geodesic variation*.

Lemma–definition 1.15. *The vector field of geodesic variation satisfies the second-order linear differential equation, called the Jacobi equation,*

$$(1.7) \qquad \nabla_{\dot\gamma}^2 \xi + \Omega(\dot\gamma, \xi)\dot\gamma = 0.$$

Proof. Define the vector field of geodesic variation $\xi(t, \alpha)$ for all geodesics of the family $\gamma_\alpha(t)$ (with small α) as the derivative $\xi(t, \alpha) = \frac{d}{d\alpha}\gamma_\alpha(t)$. Then the fields ξ and $\dot\gamma$ commute ($\{\xi, \dot\gamma\} \equiv 0$) as partial derivatives of the map $(t, \alpha) \mapsto \gamma_\alpha(t)$. Using the properties of covariant differentiation listed above and the definition of the curvature tensor, we get

$$\nabla_{\dot\gamma}^2 \xi = \nabla_{\dot\gamma} \nabla_{\dot\gamma} \xi = \nabla_{\dot\gamma} \nabla_\xi \dot\gamma = -\Omega(\dot\gamma, \xi)\dot\gamma.$$

$\square$

Assume for a moment that the curvature is positive in all two-dimensional directions containing the velocity vector of the geodesic. A closer analysis of the Jacobi equation (or its analogy with the standard pendulum; see Proposition 1.14) shows that the normal component of the vector field of geodesic variation oscillates with t. This means that geodesics with close initial velocities on a manifold of positive curvature oscillate around each other (or locally converge); see Fig. 48a. On the other hand, negativity in sectional curvatures prompts analogy with the upside-down pendulum and implies the exponential divergence of nearby geodesics from the given one; see Fig. 48b.
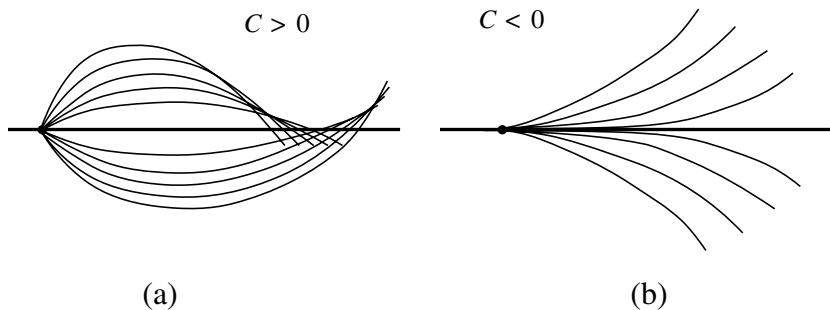
FIGURE 48.   Geodesics on manifolds of (a) positive and (b) negative curvature.

Remark 1.16. For numerical estimates of the instability, it is useful to define the *characteristic path length* as the average path length on which small errors in the initial conditions are increased by the factor of e. If the curvature of our manifold in all two-dimensional directions is bounded away from zero by the number $-b^2$, then the characteristic path length is not greater than $1/b$ (cf. Proposition 1.14). In view of the exponential character of the growth of error, the course of a geodesic line on a manifold of negative curvature is practically impossible to predict.

1.D. *Relation of the covariant and Lie derivatives*

Every vector field on a Riemannian manifold defines a differential 1-form: the pointwise inner product with vectors of the field. For a vector field v we denote by $v^\flat$ the 1-form whose value on a tangent vector at a point x is the inner product of the tangent vector with the vector $v(x)$.

Every vector field also defines a flow, which transports differential forms. For instance, one might transport the 1-form corresponding to some vector field by means of the flow of this field and get a new differential 1-form. Infinitesimally this transport is described by the Lie derivative of the 1-form (corresponding to the field) along the field itself, and the result is again a 1-form. This natural derivative of a 1-form is related to the covariant derivative of the corresponding vector field along itself by the following remarkable formula.

Theorem 1.17. *The Lie derivative of the one-form corresponding to a vector field on a Riemannian manifold differs from the one-form corresponding to the covariant derivative of the field along itself by a complete differential:*

$$(1.8) \qquad L_v(v^\flat) = (\nabla_v v)^\flat + \frac{1}{2} d\langle v, v \rangle.$$

Here $\langle v, v \rangle$ is the function on the manifold equal at each point x to the Riemannian square of the vector $v(x)$.

Note that this statement does not require the vector field to be divergence free.

Proof. Let w be a vector field commuting with the field v (i.e., $\{v, w\} = 0$).

First, since parallel translation is an isometry,

$$(1.9) \qquad L_a\langle b, c\rangle = \langle \nabla_a b, c\rangle + \langle b, \nabla_a c\rangle,$$

for any vector fields a, b, and c. Applying this to the fields $a = w$, $b = c = v$, gives $L_w\langle v, v\rangle = \langle \nabla_w v, v\rangle + \langle v, \nabla_w v\rangle$. From this we find that

$$(1.10) \qquad \langle \nabla_w v, v\rangle = \frac{1}{2}(d\langle v, v\rangle)(w).$$

Applying the isometry property (1.9) once more to $a = c = v$, $b = w$, we get

$$(1.11) \qquad L_v\langle w, v\rangle = \langle \nabla_v w, v\rangle + \langle w, \nabla_v v\rangle.$$

On the other hand, for commuting fields v and w, property (4) of Remark 1.10 implies

$$(1.12) \qquad \langle \nabla_v w, v\rangle = \langle \nabla_w v, v\rangle.$$

Substituting this into (1.11) we obtain that

$$L_v\langle w, v\rangle = \langle \nabla_w v, v\rangle + \langle w, \nabla_v v\rangle.$$

Using (1.10), we rewrite the above in the form

$$(1.13) \qquad L_v\langle w, v\rangle = \langle \nabla_v v, w\rangle + \frac{1}{2}(d\langle v, v\rangle)(w).$$

Next, we use the identity

$$(1.14) \qquad L_\xi(v^\flat(w)) = (L_\xi(v^\flat))(w) + v^\flat(L_\xi w),$$

which expresses the naturalness of the Lie derivative: The flow of ξ transports the 1-form $v^\flat$, the vector w, and the value of the 1-form on this vector. (This is the reason why vector fields are transported in the opposite direction in the definition of the Lie derivative.)

Applying (1.14) to $\xi = v$, we obtain

$$(1.15) \qquad L_v(v^\flat(w)) = (L_v(v^\flat))(w),$$

since $L_v w = -\{v, w\} = 0$. (Here we use the commutativity of v and w for the second time.)

Note that by the definition of the map $v \mapsto v^\flat$ one has

$$L_v(v^\flat(w)) = L_v\langle v, w\rangle.$$

Combining this with (1.15) and (1.13), we find that for any vector of a field w commuting with the field v,

$$(1.16) \qquad (L_v(v^\flat))(w) = (\nabla_v v)^\flat(w) + \frac{1}{2}(d\langle v, v\rangle)(w).$$

At every point x where the vector field v is nonzero, it is easy to construct a field w commuting with v and having at this point any value. Hence the identity

(1.16) implies

$$L_v(v^\flat) = (\nabla_v v)^\flat + \frac{1}{2}d\langle v, v\rangle,$$

which proves the theorem.

At the singular points $v(x) = 0$ there is nothing to prove, since both sides of the relation (1.8) are equal to zero. $\square$

Remark 1.18. One can take an arbitrary field w instead of the one commuting with v and then the formulas are slightly longer. Two commutator terms have to be introduced at the two places where commutativity was used: The additional term $\langle\{w, v\}, v\rangle$ on the right-hand side of (1.12) cancels with the extra term $\langle v, -\{v, w\}\rangle$ on the right-hand side of (1.15).

This theorem explains the form of the Euler equation of an incompressible fluid on an arbitrary Riemannian manifold M presented in Sections I.5 and I.7:

Corollary 1.19. *The Euler equation*

$$\frac{\partial v}{\partial t} = -\nabla_v v - \nabla p$$

on the Lie algebra $\mathfrak{g} = S\,\mathrm{Vect}(M)$ *of divergence-free vector fields is mapped by the inertia operator* $A : \mathfrak{g} \to \mathfrak{g}^*$ *to the Euler equation*

$$(1.17) \qquad\qquad \frac{\partial[u]}{\partial t} = -L_v[u]$$

on the dual space $\mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$ *of this algebra. Here the field* v *and the 1-form* u *are related by means of the Riemannian metric:* $u = v^\flat$, *and* $[u] \in \Omega^1/d\Omega^0$ *is the coset of the form* u.

Proof. The inertia operator $A : S\,\mathrm{Vect}(M) \to \Omega^1/d\Omega^0$ sends a divergence-free field v to the 1-form $u = v^\flat$ considered up to the differential of a function. By the above theorem, it also sends the covariant derivative $\nabla_v v$ to the Lie derivative $L_v u$ modulo the differential of another function. Hence the Euler equation for the 1-form u assumes the form

$$\frac{\partial u}{\partial t} = -L_v u - df,$$

with the function $f = p - \frac{1}{2}\langle v, v\rangle$. It is equivalent to equation (1.17) for the coset $[u]$. $\square$

§2. Sectional curvatures of Lie groups equipped with a one-sided invariant metric

Let G be a Lie group whose left-invariant metric is given by a scalar product $\langle\ ,\ \rangle$ in the Lie algebra. The sectional curvature of the group G at any point is

determined by the curvature at the identity (since by definition, left translations map the group to itself isometrically). Hence, it suffices to describe the curvatures for the two-dimensional planes lying in the Lie algebra $\mathfrak{g} = T_e G$.

Theorem 2.1 [Arn4]. *The curvature of a Lie group G in the direction determined by an orthonormal pair of vectors ξ, η in the Lie algebra $\mathfrak{g}$ is given by the formula*

$$(2.1) \qquad C_{\xi\eta} = \langle \delta, \delta \rangle + 2\langle \alpha, \beta \rangle - 3\langle \alpha, \alpha \rangle - 4\langle B_\xi, B_\eta \rangle,$$

where

$$2\delta = B(\xi, \eta) + B(\eta, \xi), \qquad 2\beta = B(\xi, \eta) - B(\eta, \xi)$$

$$(2.2) \qquad 2\alpha = [\xi, \eta], \qquad 2B_\xi = B(\xi, \xi), \qquad 2B_\eta = B(\eta, \eta),$$

and where [,] *is the commutator in $\mathfrak{g}$, and B is the bilinear operation on $\mathfrak{g}$ defined by the relation $\langle B(u, v), w \rangle = \langle u, [v, w] \rangle$ (see Chapter I).*

Remark 2.2. In the case of a two-sided invariant metric, the formula for the curvature has the particularly simple form

$$(2.3) \qquad C_{\xi\eta} = \frac{1}{4} \langle [\xi, \eta], [\xi, \eta] \rangle.$$

In particular, in this case the sectional curvatures are nonnegative in all two-dimensional directions.

Remark 2.3. The formula for the curvature of a group with a right-invariant Riemannian metric coincides with the formula in the left-invariant case. In fact, a right-invariant metric on a group is a left-invariant metric on the group with the reverse multiplication law ($g_1 \star g_2 = g_2 g_1$). Passage to the reverse group changes the signs of both the commutator and the operation B in the algebra. But in every term of the curvature formula there is a product of two operations changing the sign. Therefore, the formula for curvature is the same in the right-invariant case. The right-hand side of the Euler equation changes sign under passage to the right-invariant case.

The mapping of the group G to itself, sending each element g to the inverse element g^{-1}, is an involution preserving the identity element. It sends any left-invariant metric on the group to the corresponding right-invariant metric (defined on the Lie algebra by the same quadratic form). Hence the group with the left-invariant metric is isometric to the same group with the corresponding right-invariant metric.

To give the coordinate expression for the curvature, choose an orthonormal basis $e_1, \ldots, e_n$ for the left-invariant vector fields. The Lie algebra structure can be described by an $n \times n \times n$ array of structure constants α_{ijk} where $[e_i, e_j] = \sum_k \alpha_{ijk} e_k$, or, equivalently, $\alpha_{ijk} = \langle [e_i, e_j], e_k \rangle$. This array is skew-symmetric in the first two indices. Then Theorem 2.1 claims the following:

Theorem 2.1′(see [Mil4]). *The sectional curvature $C_{e_1e_2}$ is given by the formula*

(2.4)
$$C_{e_1e_2} = \sum_k \left(\frac{1}{2}\alpha_{12k} (-\alpha_{12k} + \alpha_{2k1} + \alpha_{k12}) \right)$$
$$- \frac{1}{4} (\alpha_{12k} - \alpha_{2k1} + \alpha_{k12})(\alpha_{12k} + \alpha_{2k1} - \alpha_{k12}) - \alpha_{k11}\alpha_{k22}.$$

Remark 2.4. Before proving the theorem we give here an account of noteworthy facts about left-invariant metrics on Lie groups that can be formulated in a coordinate-free way (and some have infinite-dimensional counterparts). We refer to [Mil4] for all the details.

- If ξ belongs to the center of a Lie algebra, then for every left-invariant metric, the inequality $C_{\xi\eta} \geq 0$ is satisfied for all η (cf. Section VI.1.A on the Virasoro algebra).
- If a connected Lie group G has a left-invariant metric with all sectional curvatures $C \leq 0$, then it is solvable (example: affine transformations of the line; see Section 1.A).
- If G is unimodular (i.e., the operators ad_u are traceless for all $u \in \mathfrak{g}$), then any such metric with $C \leq 0$ must actually be flat ($C \equiv 0$) (cf. Remark II.4.14).
- Every compact Lie group admits a left-invariant (and in fact, a bi-invariant) metric such that all sectional curvatures satisfy $C \geq 0$ (cf. Remark 2.2 above).
- If the Lie algebra of G contains linearly independent vectors ξ, η, ζ satisfying $[\xi, \eta] = \zeta$, then there exists a left-invariant metric so that the Ricci curvature $r(\xi)$ is strictly negative, while the Ricci curvature $r(\zeta)$ is strictly positive. For instance, one can define a metric on $SO(3)$, the configuration space of a rigid body, such that a certain Ricci curvature is negative!
- (Wallach) If the Lie group G is noncommutative, then it possesses a left-invariant metric of strictly negative scalar curvature.
- If G contains a compact noncommutative subgroup, then G admits a left-invariant metric of strictly positive scalar curvature [Mil4].

Proof of Theorem 2.1. To obtain explicit formulas for sectional curvatures of the group G we start by expressing the covariant derivative in terms of the operation B (or of the array α_{ijk}).

Lemma 2.5. *Let ξ and η be two left-invariant vector fields on the group G. Then the vector field $\nabla_\xi \eta$ is also left-invariant and at the point $e \in G$ is given by the following formula:*

(2.5)
$$\nabla_\xi \eta \big|_e = \frac{1}{2}([\xi, \eta] - B(\xi, \eta) - B(\eta, \xi)),$$

where on the right-hand side, ξ and η are vectors in $\mathfrak{g} = T_eG$ defining the corresponding left-invariant fields on G.

In coordinates,

$$(2.5') \qquad \nabla_{e_i} e_j = \sum_k \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij})e_k.$$

Proof of Lemma. Parallel transport on a Riemannian manifold preserves the inner product $\langle a, b \rangle$. On the other hand, the left-invariant product $\langle a, b \rangle$ of any left-invariant fields a and b is constant. Therefore, for any c, the operator ∇_c is antisymmetric on left-invariant fields:

$$\langle \nabla_c a, b \rangle + \langle \nabla_c b, a \rangle = 0.$$

Furthermore, for the covariant derivative on a Riemannian manifold the following "symmetry" condition holds (see Remark 1.10):

$$\nabla_c a - \nabla_a c = \{c, a\}.$$

Recall that on a Lie group, the Poisson bracket $\{ , \}$ of the left-invariant vector fields a and c coincides at $e \in G$ with the commutator $[,]_{\mathfrak{g}}$ in the Lie algebra:

$$(2.6) \qquad \{a, c\}\big|_e = [a, c]_{\mathfrak{g}}.$$

The Poisson bracket of two right-invariant vector fields has the opposite sign (see Remark I.2.13 or [Arn4]).

Combining the above identities, we obtain the formula

$$\langle \nabla_\xi \eta, \zeta \rangle = \frac{1}{2}(\langle [\xi, \eta], \zeta \rangle - \langle [\eta, \zeta], \xi \rangle + \langle [\zeta, \xi], \eta \rangle),$$

easily seen to be equivalent to (2.5). For the orthonormal basis $e_1, \ldots, e_n$, it immediately implies

$$\langle \nabla_{e_i} e_j, e_k \rangle = \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}).$$

This completes the proof of Lemma 2.5.   □

Finally, the coordinate expression (2.4) for sectional curvature is a straightforward consequence of formulas (1.4), (2.5'), and (2.6). Theorem 2.1' is proved.
                                                                          □

Remark 2.6. Lemma 2.5 is deduced in [Arn4] from the Euler equation $\dot{\xi} = B(\xi, \xi)$ (see Chapter I).

Consider a neighborhood of the point 0 in the Lie algebra $\mathfrak{g}$ as a chart of a neighborhood of the unit element e in the group using the exponential map $\exp : \mathfrak{g} \to G$. It sends $t\xi$ to the element $\exp(t\xi)$ of the one-parameter group starting at $t = 0$ from e with initial velocity ξ. (We leave aside the difficulties of this approach

in the infinite-dimensional case, where the image of the exponential mapping does not contain the neighborhood of the group unit element.) The exponential mapping identifies the tangent spaces of the group $T_g G$ with the Lie algebra $\mathfrak{g}$.

The Euler equation implies that the geodesic line on the group has, in our coordinates, the following expansion in t:

$$\gamma(0, \xi, t) = t\xi + \frac{t^2}{2} B(\xi, \xi) + \mathcal{O}(t^3),\ t \to 0.$$

Then the approximate translation

$$P_{\gamma(\xi,t)}\eta = \frac{1}{t} \frac{d}{d\tau}\bigg|_{\tau=0} \gamma(0, \xi + \eta\tau, t) \in T_{\gamma(t)}\mathfrak{g} = \mathfrak{g}$$

of a vector $\eta \in T_0\mathfrak{g} = \mathfrak{g}$ is explicitly given by

$$(2.7) \qquad P_{\gamma(\xi,t)}\eta = \eta + \frac{t}{2}(B(\xi, \eta) + B(\eta, \xi)) + O(t^2).$$

By definition, the covariant derivative (in the direction $\xi \in \mathfrak{g}$) of the left-invariant vector field on the group G determined by the vector $\eta \in \mathfrak{g}$ is

$$(2.8) \qquad \nabla_\xi \eta = \frac{d}{dt}\bigg|_{t=0} P_{\gamma^{-1}(\xi,t)} L_{\gamma(\xi,t)}\eta,$$

where on the left-hand side, η stands for the corresponding left-invariant vector field on G.

Note that for any Lie group G the operator of left translation by group elements $\exp a$ close to the identity (i.e., as $|a| \to 0$) acts on the Lie algebra $\mathfrak{g}$ (considered as a chart of the group) as follows:

$$(2.9) \qquad L_a\xi = \xi + \frac{1}{2}[a, \xi] + O(a^2).$$

Indeed, the general case of any Lie group follows from the calculus on matrix groups:

$$\exp a \cdot \exp b = \exp\left(a + b + \frac{1}{2}[a, b] + O(a^2) + O(b^2)\right)$$

for any Lie algebra elements $a, b \to 0$. Setting $b = \xi t,\ t \to 0,\ |a| \to 0$, we get

$$\exp a \cdot \exp \xi t = \exp\left[a + \left(\xi + \frac{1}{2}[a, \xi] + O(a^2)\right)t + O(t^2)\right],$$

which is equivalent to (2.9).

Now, substituting into (2.8) the expressions (2.7) for P_γ and (2.9) for the left translation $L_\gamma \eta$, we get the following:

$$\nabla_\xi \eta = \frac{d}{dt}\bigg|_{t=0} P_{\gamma^{-1}(\xi,t)}\left(\eta + \frac{t}{2}[\xi, \eta] + \mathcal{O}(t^2)\right)$$

$$= \frac{d}{dt}\bigg|_{t=0}\left(\eta + \frac{t}{2}([\xi, \eta] - B(\xi, \eta) - B(\eta, \xi)) + \mathcal{O}(t^2)\right).$$

$\square$

Theorem 2.1 can now be proven in the following coordinate-free way (see [Arn4]).

Proof of Theorem 2.1. Let ξ, η be left-invariant vector fields on the group G. Then the fields $\{\xi, \eta\}$, $\nabla_\xi \eta$, and $\nabla_\eta \xi$ are left-invariant as well. The formula (2.5), combined with the notations (2.2), gives the following values of these vector fields at the identity $\mathrm{Id} \in G$:

$$\nabla_\xi \xi = -2B_\xi, \qquad \nabla_\xi \eta = \alpha - \delta,$$

(2.10) $\qquad\qquad \nabla_\eta \eta = -2B_\eta, \qquad \nabla_\eta \xi = -(\alpha + \delta).$

Now, in order to evaluate the terms in (1.4), we use these expressions along with the skew symmetry of ∇ to obtain

$$\langle -\nabla_\xi \nabla_\eta \xi, \eta \rangle = \langle \nabla_\xi \eta, \nabla_\eta \xi \rangle = -\langle \alpha - \delta, \alpha + \delta \rangle,$$

(2.11) $\qquad\qquad \langle \nabla_\eta \nabla_\xi \xi, \eta \rangle = \langle \nabla_\xi \xi, \nabla_\eta \eta \rangle = -4 \langle B_\xi, B_\eta \rangle.$

Moreover, by virtue of (2.6),

$$\langle \nabla_{\{\xi,\eta\}} \xi, \eta \rangle = \langle \nabla_{[\xi,\eta]} \xi, \eta \rangle$$
$$= \frac{1}{2} \langle [[\xi, \eta], \xi], \eta \rangle - \frac{1}{2} \langle B([\xi, \eta], \xi), \eta \rangle - \frac{1}{2} \langle B(\xi, [\xi, \eta]), \eta \rangle$$

(2.12) $\qquad\qquad = -2\langle \alpha, \alpha \rangle + 2\langle \alpha, \beta \rangle.$

Finally, incorporating (2.10–2.12) into the definition (1.3–1.4) of the sectional curvature, we get

$$C_{\xi\eta} = -\langle \alpha - \delta, \alpha + \delta \rangle - 4\langle B_\xi, B_\eta \rangle - 2\langle \alpha, \alpha \rangle + 2\langle \alpha, \beta \rangle,$$

which is equivalent to (2.1). This completes the proof. $\qquad\qquad\qquad\square$

§3. Riemannian geometry of the group of area-preserving diffeomorphisms of the two-torus

3.A. *The curvature tensor for the group of torus diffeomorphisms*

The coordinate-free formulas for the sectional curvature allow one to apply them to the infinite-dimensional case of groups of diffeomorphisms. The numbers that we obtain by applying the formula for the curvature of a Lie group to these infinite-dimensional groups are naturally called the curvatures of the diffeomorphism groups. We describe in detail the case of area-preserving diffeomorphisms of the two-dimensional torus and review the results for the two-sphere (the S^2 case is of special interest because of its relation to atmospheric flows and weather predictions), for the n-dimensional torus T^n (the T^3 case is important for stability analysis of the ABC flows), for a compact two-dimensional surface, and for any flat manifold.

We start with the two-dimensional torus T^2 equipped with the Euclidean metric: $T^2 = \mathbb{R}^2/\Gamma$, where Γ is a lattice (a discrete subgroup) in the plane, e.g., the set of points with integral coordinates.

Consider the Lie algebra of divergence-free vector fields on the torus with a *single-valued stream function* (or a single-valued Hamiltonian function, with respect to the standard symplectic structure on T^2 given by the area form). The corresponding group $S_0\mathrm{Diff}(T^2)$ consists of area-preserving diffeomorphisms isotopic to the identity that leave the "center of mass" of the torus fixed.

Remark 3.1. The subgroup $S_0\mathrm{Diff}(T^2)$ is a *totally geodesic submanifold* of the group $S\mathrm{Diff}(T^2)$ of all area-preserving diffeomorphisms; that is, any geodesic on $S_0\mathrm{Diff}(T^2)$ is a geodesic in the ambient group. This follows from the momentum conservation law: If at the initial moment the velocity field of an ideal fluid has a single-valued stream function, then at all other moments of time the stream function will also be single-valued. (Note that a similar statement holds in the more general case of a manifold M with boundary: The two Lie subgroups in $S\,\mathrm{Diff}(M)$ corresponding to the Lie subalgebras of exact ($\mathfrak{g}_0$) and semiexact ($\mathfrak{g}_{se}$) vector fields (see Remark I.7.15) form totally geodesic submanifolds in the Lie group of all volume-preserving diffeomorphisms of M.)

The right-invariant metric on the group $S\mathrm{Diff}(T^2)$ is defined by the (doubled) kinetic energy: Its value at the identity of the group on a divergence-free vector field $v \in S\,\mathrm{Vect}(T^2)$ is $\langle v, v \rangle = \int_{T^2}(v, v)\, d^2x$. We will describe the sectional curvatures of the group $S_0\mathrm{Diff}(T^2)$ in various two-dimensional directions passing through the identity of the group. The curvatures of $S_0\mathrm{Diff}(T^2)$ and $S\mathrm{Diff}(T^2)$ in these directions are the same, since the submanifold $S_0\mathrm{Diff}(T^2)$ is totally geodesic.

The divergence-free vector fields that constitute the Lie algebra $S_0\,\mathrm{Vect}(T^2)$ of the group $S_0\,\mathrm{Diff}(T^2)$ can be described by their stream (i.e., Hamiltonian) functions with zero mean ($v = -H_y\partial/\partial x + H_x\partial/\partial y$). Thus in the sequel the set $S_0\,\mathrm{Vect}(T^2)$ will be identified with the space of real functions on the torus having average value zero. It is convenient to define such functions by their Fourier coefficients and to carry out all calculations over $\mathbb{C}$.

We now complexify our Lie algebra, inner product $\langle\ ,\ \rangle$, commutator $[\ ,\ ]$, and the operation B in the algebra, as well as the Riemannian connection and curvature tensor Ω, so that all these operations become (multi-) linear in the complex vector space of the complexified Lie algebra.

To construct a basis of this vector space we let e_k (where k, called a *wave vector*, is a point of the Euclidean plane) denote the function whose value at a point x of our plane is equal to $e^{i(k,x)}$.

This determines a function on the torus if the inner product (k, x) is a multiple of 2π for all $x \in \Gamma$. All such vectors k belong to a lattice Γ^* in $\mathbb{R}^2$, and the functions $\{e_k \mid k \in \Gamma^*, k \neq 0\}$ form a basis of the complexified Lie algebra.

Theorem 3.2 [Arn4, 16]. *The explicit formulas for the inner product, commutator, operation B, connection, and curvature of the right-invariant metric on the group $S_0 \operatorname{Diff}(T^2)$ have the following form:*

(3.1a)
$$\langle e_k, e_\ell \rangle = 0 \qquad for \quad k + \ell \neq 0,$$
$$\langle e_k, e_{-k} \rangle = k^2 S,$$

(3.1b) $[e_k, e_\ell] = (k \times \ell) e_{k+\ell}$ *where* $k \times \ell = k_1 \ell_2 - k_2 \ell_1,$

(3.1c) $B(e_k, e_\ell) = b_{k,\ell} e_{k+\ell}$ *where* $b_{k,\ell} = (k \times \ell) \dfrac{k^2}{(k+\ell)^2},$

(3.1d) $\nabla_{e_k} e_\ell = d_{\ell,k+\ell} e_{k+\ell}$ *where* $d_{u,v} = \dfrac{(v \times u)(u, v)}{v^2},$

(3.1e)
$$\Omega_{k,\ell,m,n} := \langle \Omega(e_k, e_\ell) e_m, e_n \rangle = 0 \qquad if \quad k + \ell + m + n \neq 0$$
$$\Omega_{k,\ell,m,n} = (a_{\ell n} a_{km} - a_{\ell m} a_{kn}) S \qquad if \quad k + \ell + m + n = 0,$$

where

(3.1f) $a_{uv} = \dfrac{(u \times v)^2}{|u + v|}.$

In these formulas, S is the area of the torus, and $u \times v$ the (oriented) area of the parallelogram spanned by u and v. The parentheses (u, v) denote the Euclidean scalar product in the plane, and angled brackets denote the scalar product in the Lie algebra.

We postpone the proof of this theorem, as well as of the corollaries below, until the next section. The formulas above allow one to calculate the sectional curvature in any two-dimensional direction.

Example 3.3. Consider the parallel sinusoidal steady fluid flow given by the stream function $\xi = \cos(k, x) = (e_k + e_{-k})/2$. Let η be any other real vector of the algebra $S_0 \operatorname{Vect}(T^2)$ (i.e., $\eta = \Sigma x_\ell e_\ell$ with $x_{-\ell} = \bar{x}_\ell$).

Theorem 3.4. *The curvature of the group $S_0 \operatorname{Diff}(T^2)$ in any two-dimensional plane containing the direction ξ is*

(3.2)
$$C_{\xi\eta} = -\frac{S}{4} \sum_\ell a_{k\ell}^2 |x_\ell + x_{\ell+2k}|^2,$$

and therefore nonpositive.

Corollary 3.5. *The curvature is equal to zero only for those two-dimensional planes that consist of parallel flows in the same direction as ξ, such that $[\xi, \eta] = 0$.*

Corollary 3.6. *The curvature in the plane defined by the stream functions $\xi = \cos(k, x)$ and $\eta = \cos(\ell, x)$ is*

$$(3.3) \qquad C_{\xi\eta} = -(k^2 + \ell^2) \sin^2 \alpha \cdot \sin^2 \beta/(4S),$$

where S is the area of the torus, α is the angle between k and ℓ, and β is the angle between $k + \ell$ and $k - \ell$.

Corollary 3.7. *The curvature of the area-preserving diffeomorphism group of the torus $\{(x, y) \bmod 2\pi\}$ in the two-dimensional directions spanned by the velocity fields $(\sin y, 0)$ and $(0, \sin x)$ is equal to $C = -1/(8\pi^2)$.*

Remark 3.8. These calculations show that in many directions the sectional curvature is negative, but in a few it is positive. The stability of the geodesic is determined by the curvatures in the directions of *all* possible two-dimensional planes passing through the velocity vector of the geodesic at each of its points (the Jacobi equation).

Any fluid flow on the torus is a geodesic of our group. However, calculations simplify noticeably for a stationary flow. In this case the geodesic is a one-parameter subgroup of our group. Then the curvatures in the directions of all planes passing through velocity vectors of the geodesic at all of its points are equal to the curvatures in the corresponding planes going through the velocity vector of this geodesic at the initial moment of time. To prove it, (right-) translate the plane to the identity element of the group. Thus, stability of a stationary flow depends only on the curvatures in those two-dimensional planes in the Lie algebra that contain the velocity field of the steady flow.

3.B. Curvature calculations

Proof of Theorem 3.2. Formula (3.1a) is an immediate consequence of the definition. Statement (3.1b) follows from the version of (2.6) for right-invariant fields: $\{a, c\}\big|_e = -[a, c]_{\mathfrak{g}}$. Moreover, combining (3.1b) with the definition of B, we come to the relation

$$(3.4) \qquad \langle B(e_k, e_\ell), e_m \rangle = (\ell \times m)\langle e_{\ell+m}, e_k \rangle.$$

Further, application of (3.1a) shows that $B(e_x, e_\ell)$ is orthogonal to e_m for $k + \ell + m \neq 0$. Thus, $B(e_k, e_\ell) = b_{k,\ell} e_{k+\ell}$. Expression (3.1c) follows from (3.1a) and (3.4) for $m = -k - \ell$.

Now, formulas (3.1b,c) along with expression (2.5) for the covariant derivative imply that

$$\nabla_{e_k} e_\ell = \frac{1}{2}(k \times \ell)\left(1 - \frac{k^2 - \ell^2}{(k + \ell)^2}\right) e_{k+\ell}.$$

This implies (3.1d) after the evident simplification

$$\frac{1}{2}\left(1 - \frac{k^2 - \ell^2}{(k + \ell)^2}\right) = \frac{(\ell, k + \ell)}{(k + \ell)^2}.$$

In order to find the curvature tensor (1.3), we first compute from (3.1d)

$$\nabla_{e_k}\nabla_{e_\ell} e_m = d_{\ell+m,k+\ell+m} d_{m,\ell+m} e_{k+\ell+m},$$

$$\nabla_{[e_k, e_\ell]} e_m = (k \times \ell) d_{m,k+\ell+m} e_{k+\ell+m}.$$

Along with (3.1a) this implies that $\Omega_{k,\ell,m,n} = 0$ for $k + \ell + m + n \neq 0$, and

$$\Omega_{k,\ell,m,n} = (d_{k+m,n} d_{n,k+n} - d_{\ell+m,n} d_{m,\ell+n} + (k \times \ell) d_{m,n}) n^2 S$$

for $k + \ell + m + n = 0$ (note that $d_{u,v}$ is symmetric in u and v due to (3.1d)). We leave to the reader the reduction of this identity to the form (3.1e). $\qquad\square$

Proof of Theorem 3.4. To derive formula (3.2), we substitute $\xi = \frac{e_k + e_{-k}}{2}$ and $\eta = \sum_\ell x_\ell e_\ell$ into

$$C_{\xi\eta} = \langle \Omega(\xi,\eta)\xi, \eta \rangle$$

$$= \frac{1}{4}\sum_\ell (\Omega_{k,\ell,k,-2k-\ell} x_\ell x_{-2k-\ell} + \Omega_{-k,\ell,-k,2k-\ell} x_\ell x_{2k-\ell}$$

$$+ \Omega_{k,\ell,-k,-\ell} x_\ell x_{-\ell} + \Omega_{-k,\ell,k,-\ell} x_\ell x_{-\ell}).$$

Taking into account the relations (3.1e–f) of Theorem 3.2, one obtains the coefficients of this quadratic form:

$$\Omega_{k,\ell,k,-2k-\ell} = \Omega_{-k,\ell,k,-\ell} = -a_{k,\ell}^2 S,$$

$$\Omega_{k,\ell,-k,-\ell} = \Omega_{-k,\ell,-k,2k-\ell} = -a_{k,-\ell}^2 S.$$

Then the form $C_{\xi\eta}$ for the orthonormal vectors ξ and η becomes

$$\langle \Omega(\xi, \eta)\xi, \eta \rangle = -\frac{S}{4}\sum_\ell [a_{k,\ell}^2 (x_\ell x_{-2k-\ell} + x_\ell x_{-\ell}) + a_{k,-\ell}^2 (x_\ell x_{2k-\ell} + x_\ell x_{-\ell})]$$

$$= -\frac{S}{4}\sum_\ell a_{k,\ell}^2 (x_\ell x_{-2k-\ell} + x_\ell x_{-\ell} + x_{\ell+2k} x_{-\ell} + x_{\ell+2k} x_{-\ell-2k}),$$

where the last identity is due to $a_{k,-\ell}^2 = a_{k,\ell-2k}^2$ (see (3.1f)). Finally, from the reality condition $x_{-j} = \bar{x}_j$, we get

$$C_{\xi\eta} = -\frac{S}{4}\sum_\ell a_{k,\ell}^2 (x_\ell \bar{x}_{\ell+2k} + x_\ell \bar{x}_\ell + x_{\ell+2k}\bar{x}_\ell + x_{\ell+2k}\bar{x}_{\ell+2k}),$$

which is equivalent to (3.2).   □

Proof of Corollary 3.6. By definition the sectional curvature in the plane spanned by a pair of orthogonal vectors ξ and η is

$$C_{\xi,\eta} = \frac{\langle \Omega(\xi,\eta)\xi, \eta \rangle}{\langle \xi, \xi \rangle \langle \eta, \eta \rangle}.$$

For our choice of ξ and η we have $\langle \xi, \xi \rangle = k^2 S/2$, $\langle \eta, \eta \rangle = l^2 S/2$, and $x_\ell = x_{-\ell} = \frac{1}{2}$. Therefore, by virtue of Theorem 3.4 and (3.1f), one obtains

$$\langle \Omega(\xi,\eta)\xi, \eta \rangle = -\frac{S}{8}(a_{k,\ell}^2 + a_{k,-\ell}^2).$$

Moreover, the explicit expression (3.1f) gives the identity

$$a_{k,\ell}^2 + a_{k,-\ell}^2 = (k \times \ell)^4 \left(\frac{1}{h_+^2} + \frac{1}{h_-^2} \right),$$

with $h_\pm := k \pm \ell$. This, in turn, can be written as

$$a_{k,\ell}^2 + a_{k,-\ell}^2 = \frac{(k \times \ell)^2 (h_+ \times h_-)^2}{2h_+^2 h_-^2}(k^2 + \ell^2),$$

where we made use of the evident relations $h_+^2 + h_-^2 = 2(k^2 + \ell^2)$ and $h_+ \times h_- = -2(k \times \ell)$. Putting all the above together and substituting $(k \times \ell)^2 = k^2 \ell^2 \sin^2 \alpha$, $(h_+ \times h_-)^2 = h_+^2 h_-^2 \sin^2 \beta$, we obtain formula (3.3) of the corollary.   □

§4. Diffeomorphism groups and unreliable forecasts

4.A. *Curvatures of various diffeomorphism groups*

For an arbitrary compact n-dimensional closed Riemannian manifold M, the curvatures of the diffeomorphism group $S_0 \, \text{Diff}(M)$ were calculated by Lukatsky [Luk5] (see also [Smo2]). We refer to [Luk3] for computations of the curvature tensor for the diffeomorphism group of any compact two-dimensional surface (the case of the two-dimensional sphere S^2, important for meteorological applications, can be found in [Luk1, Yo]) and to [Luk4] for those of a locally flat manifold (the case of the flat n-dimensional torus was treated in [Luk2]; see also explicit formulas for T^3 in [KNH]).

Riemannian geometry and curvatures of the semidirect product groups, relevant for ideal magnetohydrodynamics, are considered in [Ono]. In [Mis3] the curvatures of the group of all diffeomorphisms of a circle are discussed. The latter group, as well as its extension called the *Virasoro group*, is the configuration space of the Korteweg–de Vries equation; see Section VI.1.A. The geometry of bi-invariant metrics and geodesics on the symplectomorphism groups was studied in [Don].

Another way of investigating the Riemannian geometry of the group of volume-preserving diffeomorphisms is to embed it as a submanifold in the group of all

diffeomorphisms of the manifold and then to study the exterior geometry of the corresponding submanifold (see Section 5 below).

Keeping in mind applications to weather forecasting, we look first at the group $S\,\mathrm{Diff}(S^2)$ of diffeomorphisms preserving the area on a standard sphere $S^2 \subset \mathbb{R}^3 = \{x, y, z\}$. Consider the following two steady flows on S^2: the rotation field $u = (-y, x, 0)$ and the nonrealistic "tradewind current" $v = z \cdot (-y, x, 0)$; Fig. 49 (the real tradewind current has the same direction in the northern and southern hemispheres).

It was proved in [Luk1] that the sectional curvatures C_{uw} in all two-planes containing u are nonnegative for every field $w \in S\,\mathrm{Vect}(S^2)$, while for two-planes containing the field v the curvatures C_{vw} are negative for "most" directions w. Notice that the nonrealistic tradewind current v is a "spherical counterpart" of the parallel sinusoidal flow on the torus $\xi = (\sin y, 0)$ (see statements 3.4–3.7 above).
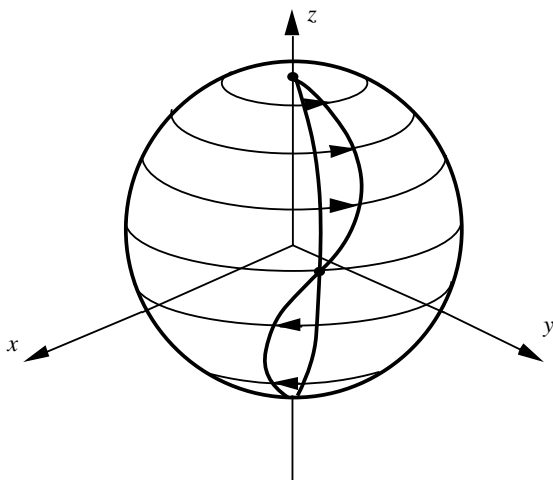
FIGURE 49.   The velocity profile of the tradewind current on the sphere.

In the case of volume-preserving diffeomorphisms of a three-dimensional domain $M \subset \mathbb{R}^3$ of Euclidean space (most important for hydrodynamics), the Jacobi equation was used by Rouchon to obtain the following information on the sectional curvatures of the diffeomorphism group.

Definition 4.1 [Rou2]. A divergence-free vector field on M is said to be a *perfect eddy* if it is equal to the velocity field of a solid rotating with a constant angular velocity around a fixed axis (in particular, the vorticity is constant). Such a fluid motion is possible if and only if the domain M admits an axis of symmetry.

Theorem 4.2 [Rou2]. *If the velocity field $v(t)$ of a flow of an ideal incompressible fluid filling a domain M is a perfect eddy with constant vorticity, then the*

sectional curvature in every two-dimensional direction in $S\,\mathrm{Vect}(M)$ containing v is nonnegative.

If the velocity $v(t)$ of an ideal fluid flow is not a perfect eddy, then for each time t there always exist plane sections containing $v(t)$ (the velocity along the geodesic) where the sectional curvature is strictly negative.

The result on nonnegativity of all the sectional curvatures holds also for rotations of spheres of arbitrary dimension [Mis1].

Remark 4.3. One can expect that the negative curvature of the diffeomorphism group causes exponential instability of geodesics (i.e., flows of the ideal fluid) in the same way as for a finite-dimensional Lie group (see, e.g., computer simulations in [KHZ]). For instance, on an n-dimensional compact manifold with nonpositive sectional curvatures the Jacobi equation along the fluid motion with constant pressure function always has an unbounded solution [Mis1].

We emphasize that the instability discussed here is the exponential instability (also called the *Lagrangian instability*) of the *motion of the fluid*, not of its velocity field (compare with Section II.4). The above result shows that from a Lagrangian point of view, all solutions of the Euler equation in $M \subset \mathbb{R}^3$ (with the exception of the perfect eddy) are unstable.

On the other hand, a stationary flow can be a Lyapunov stable solution of the Euler equation, while the corresponding motion of the fluid is exponentially unstable. The reason is that a small perturbation of the fluid velocity field can induce an exponential divergence of fluid particles. Then, even for a well-predicted velocity field (the case of a stable solution of the Euler equation), we cannot predict the motion of the fluid mass without a great loss of accuracy.

Remark 4.4. The curvature formulas for the diffeomorphism groups $S\,\mathrm{Diff}(M)$ simplify drastically for a *locally flat* (or *Euclidean*) manifold M, i.e., for a Riemannian manifold allowing local charts in which the Riemannian metric becomes Euclidean. Let $p : \mathrm{Vect}(M) \to S\,\mathrm{Vect}(M)$ be the orthogonal projection of the space of all smooth vector fields onto their divergence-free parts, where the orthogonality is considered with respect to the L^2-inner product on $\mathrm{Vect}(M)$. Let $q = \mathrm{Id} - p : \mathrm{Vect}(M) \to \mathrm{Grad}(M)$ be the orthogonal projection onto the space of the gradient vector fields.

Consider some local Euclidean coordinates $\{x_1, \ldots, x_n\}$ on M and assign to a pair of vector fields u and v their covariant derivative in M:

$$\nabla_u v := \sum_i \left(\sum_j u_j(x) \frac{\partial v_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

Theorem 4.5 [Luk4]**.** *Let M be locally Euclidean. Then the sectional curvature C_{uv} for an orthonormal pair of divergence-free vector fields $u, v \in S\,\mathrm{Vect}(M)$ is*

$$C_{uv} = -\langle q(\nabla_u v), q(\nabla_u v) \rangle + \langle q(\nabla_u u), \nabla_v v \rangle.$$

Corollary 4.6 [Luk4]. *If the vector field $\nabla_u u$ is divergence free (i.e., $q(\nabla_u u) \equiv 0$; for instance, u is a simple harmonic on T^n), then the curvature is nonpositive:*

$$C_{uv} = -\langle q(\nabla_u v), q(\nabla_u v)\rangle.$$

Remark 4.7. It is natural to describe the curvature tensor for the three-dimensional torus T^3 in the basis e_k of $S_0 \operatorname{Diff}(T^3)$, where $e_k = e^{i(k,x)}, k \in \mathbb{Z}^3\backslash\{0\}$. An arbitrary velocity field $u(x)$ is represented as $u(x) = \sum_k u_k e_k$, where the Fourier components satisfy $(k, u_k) = 0$ (divergence free) and $u_{-k} = \bar{u}_k$ (reality condition). Then, according to [NHK], one has

$$\langle \Omega(u_k e_k, v_\ell e_\ell) w_m e_m, z_n e_n\rangle$$
$$= (2\pi)^3 \left(\frac{(u_k, m)(w_m, k)}{|k+m|} \cdot \frac{(v_\ell, n)(z_n, \ell)}{|\ell+n|} - \frac{(v_\ell, m)(w_m, \ell)}{|\ell+m|} \cdot \frac{(u_k, n)(z_n, k)}{|n+k|} \right).$$

All the sectional curvatures in the three-dimensional subspace of the ABC flows in $S_0 \operatorname{Diff}(T^3)$ (see Section II.1) are equal to one and the same negative constant; i.e., the curvatures do not depend on A, B, and C [KNH].

Fix a divergence-free vector field $v \in S\operatorname{Vect}(T^k)$ on the k-dimensional flat torus T^k. The average of the sectional curvatures of all tangential 2-planes in $S\operatorname{Vect}(T^k)$ containing v is characterized by an infinite-dimensional analogue of the normalized Ricci curvature.

Definition 4.8. Let Δ be the Laplace–Beltrami operator on vector fields from $S\operatorname{Vect}(T^k)$, and $\{e_i \mid i = 1, 2, \dots\}$ be its orthonormal ordered $(-\lambda_i \leq -\lambda_{i+1})$ eigenbasis $(\Delta e_i = \lambda_i e_i)$. Define the *normalized Ricci curvature* in the direction v by

$$\operatorname{Ric}(v) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} C_{ve_i}.$$

The normalized Ricci curvature in a given direction on a finite-dimensional manifold is the average of the sectional curvatures of all tangential 2-planes containing the direction (Definition 1.13). It differs from the classical Ricci curvature by the factor of *(dimension of manifold)*-1, and it makes sense as the dimension goes to infinity.

Theorem 4.9 [Luk2]. *For a divergence-free vector field $v \in S\operatorname{Vect}(T^k)$ on the flat torus T^k,*

$$\operatorname{Ric}(v) = -\frac{k+1}{\operatorname{vol}(T^k) \cdot (k-1)k(k+2)} \|\sqrt{-\Delta}v\|_{L^2(T^k)}^2,$$

where $\sqrt{-\Delta}$ is the "positive" square root of the minus Laplace operator on vector fields.

4.B. Unreliability of long-term weather predictions

To apply the curvature calculations above, we make the following simplifying assumption: The atmosphere is a two-dimensional homogeneous incompressible fluid over the *two-torus*, and the motion of the atmosphere is approximately a "tradewind current" parallel to the equator of the torus and having a sinusoidal velocity profile.

Though the *two-sphere* is a better approximation for the earth than the two-torus, the calculations carried out for a "tradewind current" over S^2 in [Luk1] show the same order of magnitude for curvatures in both groups $S_0 \, \mathrm{Diff}\,(T^2)$ and $S \, \mathrm{Diff}\,(S^2)$. Hence, the same conclusions on the characteristic path length and instability of flows hold in both cases.

To estimate the curvature, we consider the "tradewind current" with velocity field $\xi(x, y) = (\sin y, 0)$ on the torus $T^2 = \{(x, y) \mod 2\pi\}$. Then, Theorem 3.4 shows that the curvature of the group $S_0 \, \mathrm{Diff}\,(T^2)$ in the planes containing ξ (with the wave vector $k = (0, 1)$) varies within the limits $-2/S < C < 0$, where $S = 4\pi^2$ is the area of the torus. However, the lower limit here is obtained by a rather crude estimate. To make a rough estimate of the characteristic path length, we take a quarter of this limit as the value of the "mean curvature" $C_0 = -1/(2S)$. There exist many two-dimensional directions with curvatures of approximately this size.

Having agreed to start with this value C_0 for the curvatures, we obtain the characteristic path length $s = 1/\sqrt{-C_0} = \sqrt{2S}$; see Remark 1.16. (Recall that the characteristic path length is the average path length on which a small error in the initial condition grows by the factor of e.) Note that along the group $S \, \mathrm{Diff}\,(T^2)$, the velocity of motion corresponding to the "tradewind current" ξ is equal to $\|\xi\|_{L^2(T^2)} = \sqrt{S/2}$ (since the average square value of the sine is $1/2$). Hence, the time it takes for our flow to travel the characteristic path length is equal to 2.

Now, the fastest fluid particles go a distance of 2 after this time, i.e., $1/\pi$ part of the entire orbit around the torus. Thus, if we take our value of the mean curvature, then the error grows by $e^\pi \approx 20$ after the time of one orbit of the fastest particle. Taking 100 km/hour as the maximal velocity of the tradewind current, we get 400 hours for the time of orbit, i.e., less than three weeks.

Thus, if at the initial moment, the state of the weather was known with small error ε, then the order of magnitude of the error of prediction after n months would be

$$10^{kn}\varepsilon, \qquad \text{where} \quad k = \frac{30 \cdot 24}{400} \log_{10}(e^\pi) \approx 2.5.$$

For example, to predict the weather two months in advance we must have five more digits of accuracy than the prediction accuracy. In practice, this implies that calculating the weather for such a period is impossible.

§5. Exterior geometry of the group of volume-preserving diffeomorphisms

The group $S\,\mathrm{Diff}(M)$ of volume-preserving diffeomorphisms of a Riemannian manifold M^n can be thought of as a subgroup of a larger object: the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M (cf. [E-M]). Just like its subgroup, the larger group is also equipped with a weak Riemannian metric (which is, however, no longer right-invariant):

$$(5.1) \qquad \langle g_*\xi, g_*\eta \rangle = \int_M (\xi, \eta)_{g(x)} \mu(x),$$

where $\xi, \eta \in \mathrm{Vect}(M)$; $(\xi, \eta)_a$ is the inner product of ξ and η with respect to the metric (,) on M at the point a; and $g \in \mathrm{Diff}(M)$.

Viewing the group of volume-preserving diffeomorphisms as a subgroup in the group of all diffeomorphisms of the manifold happens to be quite fruitful for various applications. To some extent the bigger group is "always flatter" than the subgroup. The source of many simplifications lies in the following fact.

Theorem 5.1 [Mis1]. *The components of the curvature tensor Ω of the bigger group $\mathrm{Diff}(M)$ are the "mean values" of the curvature tensor components for the Riemannian manifold M itself:*

$$\langle \Omega(u, v)w, z \rangle = \int_M \langle \Omega_x^M (u(x), v(x))w(x), z(x) \rangle_x \mu(x),$$

where Ω_x^M is the curvature tensor of M at $x \in M$; the volume form μ is defined by the metric; and $u, v, w, z, \in \mathrm{Vect}(M)$.

Below we derive (following [Mis1, Shn4, Tod]) the second fundamental form of the embedding of the "curved" subgroup $S\,\mathrm{Diff}(M) \subset \mathrm{Diff}(M)$ into the "flatter" ambient group. Though not intrinsic in nature, it gives a nice shortcut to calculations of the curvatures.

For simplicity, let the manifold M be the flat n-torus T^n. Represent a diffeomorphism $g \in \mathrm{Diff}(T^n)$ close to the identity in the form $g(x) = x + \xi(x)$.

Proposition 5.2. *In the coordinates $\{\xi\}$, a C^1-small neighborhood of the identity $\mathrm{Id} \in \mathrm{Diff}(T^n)$ of the group $\mathrm{Diff}(T^n)$ equipped with the metric (5.1) is isometrically embedded in the Hilbert space $\mathbb{H} = \{\xi \in L^2(T^n, \mathbb{R}^n)\}$.*

Proof. The proof of Proposition 5.2 is a straightforward calculation.   □

Abusing notation, we will denote by $\mathbb{H}$ the (pre-) Hilbert space of smooth maps from the torus T^n to $\mathbb{R}^n$ equipped with the L^2 inner product. Then a neighborhood of the identity of the group $\mathrm{Diff}(T^n)$ is isometric with a neighborhood of the origin

in $\mathbb{H}$. The group $S\,\mathrm{Diff}(T^n)$ of volume-preserving diffeomorphisms of T^n will be viewed as a submanifold $\mathcal{D}$ of $\mathbb{H}$ (Fig. 50):

$$\mathcal{D} = S\,\mathrm{Diff}(T^n) = \{\xi \in L^2(T^n, \mathbb{R}^n) \mid \det\left[\mathrm{Id} + \frac{\partial \xi}{\partial x}\right] \equiv 1\} \subset \mathbb{H}.$$
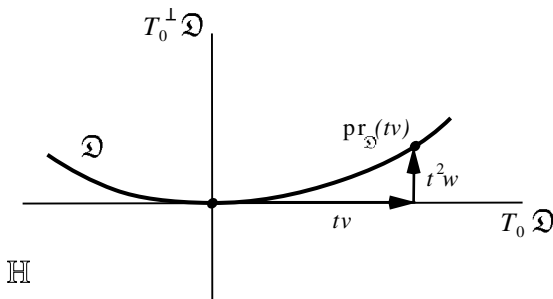

FIGURE 50.   The embedding of the volume-preserving diffeomorphisms $\mathcal{D} = S\mathrm{Diff}(M)$ into the group of all diffeomorphisms $\mathbb{H} = \mathrm{Diff}(M)$.

Definition 5.3. The *second fundamental form* L (at $0 \in \mathcal{D}$) of the embedding $\mathcal{D} \subset \mathbb{H}$ is a quadratic map $L : T_0\mathcal{D} \to T_0^{\perp}\mathcal{D}$ from the tangent space $T_0\mathcal{D} \subset \mathbb{H}$ to its orthogonal complement $T_0^{\perp}\mathcal{D} \subset \mathbb{H}$. The value of the second fundamental form $L(v, v)$ at a vector $v \in T_0\mathcal{D}$ is equal to the acceleration of a point moving by inertia along $\mathcal{D}$ with initial velocity v (see [K-N]).

In other words, L measures (the second derivative of) the "distance" in $\mathbb{H}$ between the point tv moving in the tangent space $T_0\mathcal{D}$ with constant velocity v and the orthogonal projection $\mathrm{pr}_{\mathcal{D}}$ of this point to $\mathcal{D}$:

$$(5.2) \qquad \mathrm{pr}_{\mathcal{D}}(tv) = tv + \frac{t^2}{2}L(v, v) + O(t^3) \qquad \text{as} \quad t \to 0.$$

The spaces $T_0\mathcal{D}$ and $T_0^{\perp}\mathcal{D}$ are more explicitly described as follows

$$T_0\mathcal{D} = S\,\mathrm{Vect}(T^n) = \{v \in \mathrm{Vect}(T^n) \mid \mathrm{div}\, v = 0\},$$

$$T_0^{\perp}\mathcal{D} = \mathrm{Grad}(T^n) = \{w \in \mathrm{Vect}(T^n) \mid w = \nabla p, \text{ for some } p \in C^{\infty}(T^n)\}$$

(we have included in $T_0\mathcal{D}$ the divergence-free fields shifting the center of mass of T^n, and hence $T_0^{\perp}\mathcal{D}$ consists of the gradients of all univalued functions).

Observe that for a vector field $v \in S\,\mathrm{Vect}(T^n)$ the transformation $x \mapsto x+tv(x)$ means that every point $x \in T^n$ moves uniformly with velocity $v(x)$ along the straight line passing through x. Such transformations are diffeomorphisms for smooth $v(x)$ and sufficiently small $t > 0$.

To demonstrate the machinery, we confine ourselves, for now, to the case $n = 2$ and give an alternative proof of Theorem 3.4 on curvatures $S\,\mathrm{Diff}_0(T^2)$ in two-dimensional directions containing the sinusoidal flow ξ on the torus.

Proof of Theorem 3.4. A vector field $v \in S \operatorname{Vect}(T^2)$ can be described by the corresponding (univalued) stream (or Hamiltonian) function ψ: $v = \operatorname{sgrad} \psi$; that is, $v_1 = -\partial\psi/\partial x_2$ and $v_2 = \partial\psi/\partial x_1$.

Theorem 5.4. *The value of the second fundamental form L at a vector field v is*

$$(5.3) \qquad L(v, v) = -2\nabla(\Delta^{-1}(\det[\operatorname{Hess}\psi])),$$

where $\operatorname{Hess}\psi$ *is the Hessian matrix of the stream function ψ of the field v, and Δ^{-1} is the Green operator for the Laplace operator Δ in the class of functions with zero mean on T^2.*

Proof of Theorem 5.4. The following evident relation shows how far the transformation $\operatorname{Id} + tv$ is from $\mathcal{D}$ (i.e., from being volume-preserving):

$$\det\left[\operatorname{Id} + t\frac{\partial v}{\partial x}\right] = 1 + t \operatorname{div} v + t^2(v_{1,1}v_{2,2} - v_{1,2}v_{2,1}) = 1 + t^2(v_{1,1}v_{2,2} - v_{1,2}v_{2,1}),$$

where $v_{i,j} := \partial v_i/\partial x_j$, and the last equality is due to $\operatorname{div} v = 0$. The transformation $\operatorname{Id} + tv$ does not belong to $\mathcal{D}$, and it changes the standard volume element on T^2 by a term quadratic in t.

Hence, we have to adjust tv by adding to it a vector field $t^2w \in T_0^{\perp}\mathcal{D}$ to suppress the divergence of $\operatorname{Id} + tv$. To compute the second fundamental form (see (5.2)) observe that its value at the vector v is $L(v, v) = 2w$, where $w \in T_0^{\perp}\mathcal{D}$ is defined by the condition that the transform $x \mapsto x + tv + t^2w$ is volume-preserving modulo $O(t^3)$.

The defining relation on the field w: $\operatorname{div} w = -(v_{1,1}v_{2,2} - v_{1,2}v_{2,1})$ follows immediately from the expansion

$$\det\left(\operatorname{Id} + t\frac{\partial v}{\partial x} + t^2\frac{\partial w}{\partial x}\right) = 1 + t^2(v_{1,1}v_{2,2} - v_{1,2}v_{2,1} + \operatorname{div} w) + O(t^3).$$

From the definition of $T_0^{\perp}\mathcal{D}$, the vector field w is a gradient: $w = \nabla\varphi$. Therefore, $\operatorname{div} w = \nabla^2\varphi = \Delta\varphi$, and

$$L(v, v) = 2w = -2\nabla(\Delta^{-1}(v_{1,1}v_{2,2} - v_{1,2}v_{2,1})).$$

The introduction of the stream function ψ for the field v reduces the latter formula to the required form (5.3). $\qquad\square$

The *symmetric fundamental form* $L(u, v)$ can now be obtained from the quadratic form $L(v, v)$ via polarization:

$$L(u, v) = (L(u + v, u + v) - L(u, u) - L(v, v))/2.$$

Finally, the sectional curvature C_{uv} of the group $\mathcal{D}$ in the two-dimensional direction spanned by any two orthonormal vectors u and v is, according to the Gauss–Codazzi formula (Proposition VII.4.5 in [K-N]), given by

$$C_{uv} = \langle L(u, u), L(v, v)\rangle - \langle L(u, v), L(u, v)\rangle,$$

where $\langle\ ,\ \rangle$ is the inner product in the Hilbert space $\mathbb{H}$ (cf. also Theorem 4.5 above). For nonorthonormal vectors one has to normalize the curvature according to formula (1.5).

Example 5.5. We will now calculate (using the second fundamental form) the sectional curvature in the direction spanned by the vector fields u and v with the stream functions $\phi = \cos ay$ and $\psi = \cos bx$ (where the wave vectors of Corollary 3.6 are $k = (0, a)$ and $\ell = (b, 0)$).

One easily obtains that $\det[\text{Hess }\phi] = \det[\text{Hess }\psi] \equiv 0$, while

$$\det[\text{Hess}(\phi + \psi)] = a^2 b^2 \cos ay \cos bx.$$

Then the application of Δ^{-1} is equivalent to the multiplication of the above function by $-1/(a^2 + b^2)$. Passing to the gradient, one sees that $L(u, v)$ is the vector field

$$L(u, v) = -\frac{a^2 b^2}{a^2 + b^2}\left((b \sin bx \cos ay)\frac{\partial}{\partial x} + (a \cos bx \sin ay)\frac{\partial}{\partial y}\right),$$

while both $L(u, u)$ and $L(v, v)$ vanish. Note also that $\langle u, u \rangle = a^2 S/2$, $\langle v, v \rangle = b^2 S/2$, where S is the area of the torus. Finally, evaluating the L^2-norm of the vector field $L(u, v)$ over the torus, we come to the following formula for the sectional curvature:

$$C_{uv} = -\frac{\langle L(u, v), L(u, v)\rangle}{\langle u, u\rangle \cdot \langle v, v\rangle} = -\frac{a^2 b^2}{(a^2 + b^2)S},$$

which is in a perfect matching with (3.3).

We leave it to the reader to check that the substitution of the vector fields u and v with the stream functions $\xi = \frac{e_k + e_{-k}}{2}$ and $\eta = \Sigma x_\ell e_\ell$ (with $x_{-\ell} = \bar{x}_\ell$) into the curvature formula gives an alternative proof of Theorem 3.4 in full generality.

$\square$

Remark 5.6. Bao and Ratiu [B-R] have studied the *totally geodesic* (or *asymptotic*) directions on the Riemannian submanifold $S\,\text{Diff}(M) \subset \text{Diff}(M)$, i.e., those directions in the tangent spaces $T_g S\,\text{Diff}(M)$ (alternatively, divergence-free vector fields on M) for which the second fundamental form L of $S\,\text{Diff}(M)$, relative to $\text{Diff}(M)$, vanishes. For an arbitrary manifold M^n, they obtained an explicit description of such directions $g_* v \in T_g S\,\text{Diff}(M)$ in the form of a certain first-order nonlinear partial differential equation on v. For the two-dimensional case this equation can be rewritten as an equation for the stream function. Assume for simplicity that $\partial M = \emptyset$ and $H^1(M) = 0$.

Theorem 5.7 [B-R]. *For a two-dimensional Riemannian manifold M the divergence-free vector field $g_* v \in T_g S\,\text{Diff}(M)$ is a totally geodesic direction*

on $S\mathrm{Diff}(M)$ if and only if the stream function ψ of the field v satisfies the degenerate Monge–Ampère equation

$$\det[\mathrm{Hess}\,\psi] = \frac{K \cdot m \cdot \|\nabla\psi\|^2}{2},$$

where m is the determinant of the metric on M in given coordinates $\{x_1, x_2\}$, Hess is the Hessian matrix of ψ in these coordinates, and K is the Gaussian curvature function on M.

Their paper also contains examples of manifolds for which the Monge–Ampère equation has, or has no, solutions (see also [BLR] for a characterization of all manifolds M for which the asymptotic directions are harmonic vector fields).

Asymptotic directions on $S\mathrm{Diff}(M^2)$ arise intrinsically in the context of a discrete version of the Euler equation of an incompressible fluid [MVe]. A solution of the discretized Euler equation is a recursive sequence of diffeomorphisms. The Monge–Ampère equation is the constraint on the initial condition ensuring that all the diffeomorphisms of the sequence preserve the area element on M^2.

Remark 5.8. Consider an equivalence relation on the diffeomorphism group $\mathrm{Diff}(M)$, where two diffeomorphisms are in the same class if they differ by a volume-preserving transformation. We obtain a fibration of $\mathrm{Diff}(M)$ over the space of densities (i.e., the space of positive functions on M), with the fiber isomorphic to the set of volume-preserving diffeomorphisms $S\mathrm{Diff}(M)$. Let the manifold M^n be one of the following: an n-dimensional sphere S^n, Lobachevsky space Λ^n, or Euclidean space $\mathbb{R}^n$. According to S.M. Gusein-Zade, there is no section of this $S\mathrm{Diff}(M)$-bundle over the space of densities that is invariant with respect to motions of the corresponding space M^n. However, there exists a unique connection in this bundle that is invariant with respect to motions on M^n. Parallel translation in this connection is essentially described in the proof of the Moser theorem (cf. Lemma III.3.5). The case of a two-dimensional sphere has interesting applications in cartography.

§6. Conjugate points in diffeomorphism groups

Although in "most" of the two-dimensional directions the sectional curvatures of the diffeomorphism group $S\mathrm{Diff}(T^2)$ are negative, in some directions the curvature is positive.

Example 6.1 [Arn4]. In the algebra $S\mathrm{Vect}(T^2)$ of all divergence-free vector fields on the torus $T^2 = \{(x, y) \mod 2\pi\}$ consider the plane spanned by the two stream functions $\xi = \cos(3px - y) + \cos(3px + 2y)$ and $\eta = \cos(px + y) + \cos(px - 2y)$. Then for the sectional curvature one has

$$C_{\xi\eta} = \frac{\langle \Omega(\xi, \eta)\xi, \eta\rangle}{\langle \xi, \xi\rangle\langle \eta, \eta\rangle} \to \frac{9}{8\pi^2} > 0 \qquad \text{as} \quad p \to \infty.$$

It is tempting to conjecture, by analogy with the finite-dimensional case, that positivity of curvatures is related to the existence of the conjugate points on the group $S\text{Diff}(T^2)$.

Definition 6.2. A *conjugate point* of the initial point $\gamma(0)$ along a geodesic line $\gamma(t)$, $t \in [0, \infty)$ (on a Riemannian manifold M), is the point where the geodesic line hits an infinitely close geodesic, starting from the same point $\gamma(0)$. The conjugate points are ordered along a geodesic, and the first point is the place where the geodesic line ceases to be a local minimum of the length functional.

Strictly speaking, one considers zeros of the first variation rather than the actual intersection of geodesics (see, e.g., [K-N]).

Theorem 6.3 [Mis2]. *Conjugate points exist on the geodesic in the group $S\,\text{Diff}(T^2)$ emanating from the identity with velocity $v = \text{sgrad}\,\phi$ defined by the stream function $\phi = \cos 6x \cdot \cos 2y$.*

A segment of a geodesic line is no longer the shortest curve connecting its ends if the segment contains an interior point conjugate to the initial point (Fig. 51). Indeed, the difference of lengths of a geodesic curve segment and of *any C^1 ϵ-close curve* joining the same endpoints is of order ϵ^2 (the geodesic being an extremal). The length of a *geodesic ϵ-close* to the initial one and connecting the initial point A with its conjugate point C differs from the length of the initial geodesic between A and C by a quantity of higher order, ϵ^3. The difference between BD and $BC+CD$ is of order, ϵ^2. Hence ADB is shorter than ACB.
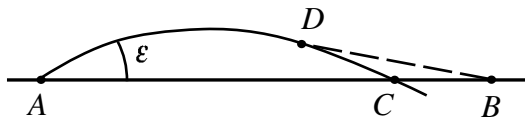
FIGURE 51. A geodesic ceases to be the length global minimum after the first conjugate point.

If a segment of a geodesic contains k interior points conjugate to the initial point, then the quadratic form of the length second variation has k negative squares.

Remark 6.4 [Mis2]. Conjugate points can be found on the group $S\,\text{Diff}(T^n)$, where T^n is a flat torus of arbitrary dimension n (this is a simple corollary of the two-dimensional example above).

More examples are provided by geodesics on the group of area-preserving diffeomorphisms of a surface of positive curvature (example: uniform rotation of the standard two-dimensional sphere; see Section 4.A).

On the other hand, those geodesics in $S\operatorname{Diff}(M)$ that are also geodesics in $\operatorname{Diff}(M)$ have no conjugate points whenever M is a Riemannian manifold of nonpositive sectional curvature [Mis1]. Such geodesics have asymptotic directions on $S\operatorname{Diff}(M)$ and correspond to the solutions of the Euler equation in M with constant pressure functions.

Remark 6.5. It has been neither proved nor disproved that the Morse index of a geodesic line corresponding to a smooth stationary flow is finite (for any finite portion of the geodesic). It is interesting to consider whether the conjugate points might accumulate in this situation. (See, however, [EbM], where the case of a flat two-dimensional torus was settled: it was shown that the exponential map on the group $S\operatorname{Diff}(T^2)$ turns out to be Fredholm of index zero, and the accumulation phenomenon does not occur.)

Note that the existence of a small geodesic segment near the initial point that is free of conjugate points follows from the nondegeneracy of the geodesic exponential map at the initial point and in its neighborhood. One might also ask what the shortest path is to a point on the geodesic that is separated from the initial point by a point conjugate to it. (One hopes that the overall picture in this classical situation is not spoiled by the pathology of the absence of the shortest path between special diffeomorphisms, discovered by Shnirelman in [Shn1]; see Section 7.D.)

§7. Getting around the finiteness of the diameter of the group of volume-preserving diffeomorphisms*

Consider a volume-preserving diffeomorphism of a bounded domain. In order to reach the position prescribed by the diffeomorphism, every fluid particle has to move along some path in the domain. The distance of this diffeomorphism from the identity is the averaged characteristic of the path lengths of the particles.

It turns out that the diameter of the group of volume-preserving diffeomorphisms of a three-dimensional ball is finite, while for a two-dimensional domain it is infinite. This difference is due to the fact that in three (and more) dimensions there is enough space for particles to move to their final places without hitting each other. On the other hand, the motion of the particles in the plane might necessitate their rotation about one another. The latter phenomenon of "braiding" makes the system of paths of particles necessarily long, in spite of the boundedness of the domain (and hence, of the distances between the initial and final positions of each particle).

In this section we describe some principal properties of the group of volume-preserving diffeomorphisms $\mathcal{D}(M) := S\operatorname{Diff}(M)$ as a metric space along with their dynamical implications.

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*This section was written by A. Shnirelman.

7.A. Interplay between the internal and external geometry of the diffeomorphism group

Definition 7.1. Let M^n be a Riemannian manifold with volume element dx. Introduce a *metric* on the group $\mathcal{D}(M) = S\,\mathrm{Diff}(M^n)$ as follows. To any path g_t, $t_1 \le t \le t_2$, on the group $\mathcal{D}(M)$ we associate its *length*:

$$(7.1) \qquad \ell\{g_t\}_{t_1}^{t_2} = \int_{t_1}^{t_2} \|\dot{g}_t\|_{L^2(M)} dt = \int_{t_1}^{t_2} \left(\int_{M^n} \left\| \frac{\partial g_t(x)}{\partial t} \right\|^2 dx \right)^{1/2} dt.$$

For two fluid configurations $f, h \in \mathcal{D}(M)$, we define the *distance* between them on $\mathcal{D}(M)$ as the infimum of the lengths of all paths connecting f and h:

$$\mathrm{dist}_{\mathcal{D}(M)}(f, h) = \inf_{\substack{\{g_t\} \subset \mathcal{D}(M) \\ g_0 = f, g_1 = h}} \ell\{g_t\}_0^1.$$

This definition makes $\mathcal{D}(M)$ into a *metric space*. Now the *diameter* of $\mathcal{D}(M)$ is the supremum of distances between its elements:

$$\mathrm{diam}(\mathcal{D}(M)) = \sup_{f, h \in \mathcal{D}(M)} \mathrm{dist}_{\mathcal{D}(M)}(f, h).$$

The metric on the group of volume-preserving diffeomorphisms $\mathcal{D}(M)$ defined here is induced by the right-invariant metric on the group defined at the identity by the kinetic energy of vector fields (compare formula (7.1) with Example I.1.3).

We start with the study of the following three intimately related problems:

A. (Diameter problem) Is the diameter of the group $\mathcal{D}(M)$ of volume-preserving diffeomorphisms infinite or finite? In the latter case, how can it be estimated for a given manifold M^n?

Let M be a bounded domain in the Euclidean space $\mathbb{R}^n$ (with the Euclidean volume element dx). In this case the diffeomorphism group $\mathcal{D}(M)$ is naturally embedded into the Hilbert space $L^2(M^n, \mathbb{R}^n)$ of vector functions on M^n. This embedding defines an isometry of $\mathcal{D}(M)$ with its (weak) Riemannian structure onto its image equipped with the Riemannian metric induced from the Hilbert space.

Definition 7.2. The *standard* distance dist_{L^2} between two diffeomorphisms $f, h \in \mathcal{D}(M) \subset L^2(M, \mathbb{R}^n)$ is the distance between them in the ambient Hilbert space $L^2(M, \mathbb{R}^n)$:

$$\mathrm{dist}_{L^2}(f, h) = \|f - h\|_{L^2(M, \mathbb{R}^n)}.$$

B. (Relation of metrics) What is the relation between the distance $\mathrm{dist}_{\mathcal{D}(M)}$ in the group $\mathcal{D}(M)$ defined above and the standard distance dist_{L^2} pulled back to $\mathcal{D}(M)$ directly from the space $L^2(M, \mathbb{R}^n)$?

Evidently, $\mathrm{dist}_{\mathcal{D}(M)} \geq \mathrm{dist}_{L^2}$. But does there exist an estimate of $\mathrm{dist}_{\mathcal{D}(M)}(g, h)$ through $\mathrm{dist}_{L^2}(g, h)$? In particular, is it true that if two volume-preserving diffeomorphisms are close in the Hilbert space, then they can be joined by a short path within the group $\mathcal{D}(M)$?

C. (Shortest path) Given two volume-preserving diffeomorphisms, does there exist a path connecting them in the group $\mathcal{D}(M)$ that has minimal length? If so, it is a geodesic; i.e., after an appropriate reparametrization it becomes a solution of the Euler equation for an ideal fluid in M^n. Finding the shortest path between two arbitrary fluid configurations promises to be a good method for constructing fluid flows.

Remark 7.3. Similar problems can be posed for diffeomorphisms of an arbitrary Riemannian manifold M^n, if we first isometrically embed M^n in $\mathbb{R}^q$ for some q. Such an embedding for which the Euclidean metric in $\mathbb{R}^q$ descends to the given Riemannian metric on M^n exists by virtue of the Nash theorem [Nash] (where one can take $q = 3n(n + 9)/2$ for a compact M^n, and $q = 3n(n + 1)(n + 9)/2$ for a noncompact M^n).

7.B. Diameter of the diffeomorphism groups

In what follows we confine ourselves to the simplest domain M^n, namely, to a unit cube: $M^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 < x_i < 1\}$. We thus avoid the topological complications due to the topology of M.

Theorem 7.4 [Shn1]. *For a unit n-dimensional cube M^n where $n \geq 3$, the diameter of the group of smooth volume-preserving diffeomorphisms $\mathcal{D}(M)$ is finite in the right-invariant metric $\mathrm{dist}_{\mathcal{D}(M)}$.*

Theorem 7.4′ [Shn5]. $\mathrm{diam}(\mathcal{D}(M^n)) \leq 2\sqrt{\frac{n}{3}}$.

These theorems generalize to the case of an arbitrary simply connected manifold M. However, the diameter can become infinite if the fundamental group of M is not trivial [ER2]. The two-dimensional case is completely different:

Theorem 7.5 [ER1, 2]. *For an arbitrary manifold M of dimension $n = 2$, the diameter of the group $\mathcal{D}(M)$ is infinite.*

One can strengthen the latter result in the following direction.

Definition 7.6. A diffeomorphism $g : M \to M$ of an arbitrary domain (or a Riemannian manifold) M is called *attainable* if it can be connected with the identity diffeomorphism Id by a piecewise-smooth path $g_t \subset \mathcal{D}(M)$ of finite length.

Theorem 7.7 [Shn2]. *Let M^n be an n-dimensional cube and $n \geq 3$. Then every element of the group $\mathcal{D}(M)$ is attainable. In the case $n = 2$, there are unattainable diffeomorphisms of the square M^2. Moreover, the unattainable diffeomorphisms can be chosen to be continuous up to the boundary ∂M^2 and identical on ∂M^2.*

A diffeomorphism g may be unattainable if its behavior near the boundary ∂M^n of M is complicated enough. We will give an example of an unattainable diffeomorphism in Section 8.A. Note that only attainable diffeomorphisms are physically reasonable, since the fluid cannot reach an unattainable configuration in a finite time.

The statement above allows one to specify Theorem 7.5:

Theorem 7.8 (= 7.5′). *For two-dimensional M, the subset of attainable diffeomorphisms in $\mathcal{D}(M)$ is of infinite diameter.*

It is not known whether all the attainable diffeomorphisms form a subgroup in $\mathcal{D}(M^2)$ (i.e., whether the inverse of an attainable diffeomorphism is attainable). It is not always true that if a path $\{g_t\} \subset \mathcal{D}(M^2)$ has finite length, then the length of the path $\{g_t^{-1}\} \subset \mathcal{D}(M^2)$ is finite. The group $\mathcal{D}(M^2)$ splits into a continuum of equivalence classes according to the following relation: Two diffeomorphisms are in the same class if they can be connected to each other by a path of finite length. Every equivalence class has infinite diameter.

The proofs of the two-dimensional results are rather transparent and are discussed in Sections 8.A–B. Various approaches to the three- (and higher-) dimensional case are, on the contrary, all quite intricate, and only the ideas are discussed below.

7.C. Comparison of the metrics and completion of the group of diffeomorphisms

The main difference between the geometries of the groups of diffeomorphisms in two and three dimensions is based on the observation that for a long path on $\mathcal{D}(M^3)$, which twists the particles in space, there always exists a "shortcut" untwisting them by "making use of the third coordinate." More precisely, the following estimate holds.

Theorem 7.9 [Shn1]. *Given dimension $n \geq 3$, there exist constants $C > 0$ and $\alpha > 0$ such that for every pair of volume-preserving diffeomorphisms $f, h \in \mathcal{D}(M^n)$ of the unit cube,*

$$\mathrm{dist}_{\mathcal{D}(M)}(f, h) \leq C(\mathrm{dist}_{L^2}(f, h))^{\alpha}.$$

Theorem 7.9′ [Shn5]. *The exponent α in this inequality is not less than $2/(n + 4)$.*

This property means that the embedding of $\mathcal{D}(M^n)$ into $L^2(M, \mathbb{R}^n)$, $n \geq 3$, is "Hölder-regular." Apparently, it is far from being smooth, i.e., $\alpha < 1$. Certainly, the Hölder property (Theorem 7.9) implies the finiteness of the diameter (Theorem 7.4).

No such estimate is true for $n = 2$. Namely, for every pair of positive constants c, C there exists a diffeomorphism $g \in \mathcal{D}(M^2)$ such that $\mathrm{dist}_{\mathcal{D}(M)}(g, \mathrm{Id}) > C$, but $\mathrm{dist}_{L^2}(g, \mathrm{Id}) < c$. This complements Theorem 7.5 but requires, of course, a separate proof.

Theorems 7.4 and 7.9 imply the following simple description of the completion of the metric space $\mathcal{D}(M^n)$ in the case $n \geq 3$.

Corollary 7.10. *For $n \geq 3$ the completion of the group $\mathcal{D}(M^n)$ in the metric $\mathrm{dist}_{\mathcal{D}(M)}$ coincides with the closure $\bar{\mathcal{D}}(M^n)$ of the group in $L^2(M^n, \mathbb{R}^n)$.*

Proof of Corollary. Each Cauchy sequence $\{g_i\}$ in $\mathcal{D}(M^n)$ (with respect to the metrics $\mathrm{dist}_{\mathcal{D}(M^n)}$) is a Cauchy sequence in $L^2(M^n, \mathbb{R}^n)$, and therefore it converges to some element $g \in L^2(M^n, \mathbb{R}^n)$.

Conversely, if g belongs to the closure of $\mathcal{D}(M^n)$ in $L^2(M^n, \mathbb{R}^n)$, then there exists a sequence of diffeomorphisms $\{g_i\} \subset \mathcal{D}(M^n)$ that converges to it in $L^2(M^n, \mathbb{R}^n)$. Therefore, by virtue of Theorem 7.9, $\{g_i\}$ is a Cauchy sequence in $\mathcal{D}(M^n)$, and thus g lies in the completion of $\mathcal{D}(M^n)$.   □

Theorem 7.11 [Shn1]. *The completion $\bar{\mathcal{D}}(M^n)$ ($n \geq 3$) of the group $\mathcal{D}(M^n)$ consists of all measure-preserving endomorphisms on M, i.e., of such Lebesgue-measurable maps $f : M^n \to M^n$ that for every measurable subset $\Omega \subset M^n$,*

$$\mathrm{mes}\, f^{-1}(\Omega) = \mathrm{mes}\, \Omega.$$

The idea for a proof of this theorem is as follows. Divide the unit cube M^n into N^n equal small cubes having linear size N^{-1}. Consider the class $\mathcal{D}_N$ of piecewise-continuous mappings, translating in a parallel way each small cube $\varkappa$ into another small cube $\sigma(\varkappa)$, where σ is some permutation of the set of small cubes. First one proves that every measure-preserving map $f : M \to M$ may be approximated with arbitrary accuracy in $L^2(M^n, \mathbb{R}^n)$ by a permutation of small cubes for sufficiently large N. In turn, every such permutation may be approximated in L^2 by a smooth volume-preserving diffeomorphism. Conversely, if a map g belongs to the closure of the diffeomorphism group $g \in \bar{D}(M^n)$, then $g = \lim g_i$, $g_i \in \mathcal{D}(M^n)$, almost everywhere in M^n, and hence g is a measure-preserving endomorphism; see [Shn1] for more detail.

In the two-dimensional case, the completion of $\mathcal{D}(M^2)$ in the metric $\mathrm{dist}_{\mathcal{D}(M)}$ is a proper subset of the L^2-closure $\bar{D}(M^2)$; no good description of this completion is known.

7.D. *The absence of the shortest path*

We will see below how the facts presented so far in this section imply a negative answer to the question of existence of the shortest path in the diffeomorphism group:

Theorem 7.12 [Shn1]. *For a unit cube M^n of dimension $n \geq 3$, there exists a pair of volume-preserving diffeomorphisms that cannot be connected within the group $\mathcal{D}(M)$ by a shortest path; i.e., for every path connecting the diffeomorphisms there always exists a shorter path.*

Thus, the attractive variational approach to constructing solutions of the Euler equations is not directly available in the hydrodynamical situation. This is not to say that the variational approach is wrong, but merely that our understanding is still incomplete and further work is required.

Remark 7.13. The proof of Theorem 7.12 is close in spirit to the Weierstrass example of a variational problem having no solution. Weierstrass proposed his example in his criticism of the use of the Dirichlet principle for proving the existence of a solution of the Dirichlet problem for the Laplace equation.
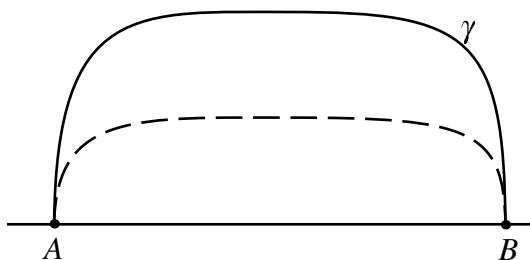
FIGURE 52.   There is no smooth shortest curve between A and B that would be orthogonal to the segment AB at the endpoints.

This example illustrates that in some cases a functional cannot attain its infimum. Consider two points A and B in the plane. We are looking for a smooth curve γ of minimal length connecting A and B such that its tangents at the points A and B are orthogonal to the line (AB); Fig. 52. It is clear that if γ is different from the segment $[AB]$, then it may be squeezed toward the line (AB) (say, by the factor $1/2$), and this transformation reduces its length. However, if γ coincides with the segment $[AB]$, it does not satisfy the boundary conditions. Thus, the infimum cannot be attained within the class of admissible curves.

Weierstrass's criticism encouraged Hilbert to establish a solid foundation for the Dirichlet variational principle.

We proceed to describe an example of a diffeomorphism $g \in \mathcal{D}(M^3)$ of the three-dimensional cube M^3 that cannot be connected to the identity diffeomorphism Id by a shortest path.

Let (x_1, x_2, z) be Cartesian coordinates in $\mathbb{R}^3$, and let $M^3 = \{0 < x_1, x_2, z < 1\}$. Consider an arbitrary diffeomorphism $g \in \mathcal{D}(M^3)$ of the form

$$g(x, z) = (h(x), z),$$

where h is an area-preserving diffeomorphism of the square M^2 and $x := (x_1, x_2)$.

Theorem 7.12′. *If*

$$\text{dist}_{\mathcal{D}(M^3)}(\text{Id}, g) < \text{dist}_{\mathcal{D}(M^2)}(\text{Id}, h),$$

then $\text{Id} \in \mathcal{D}(M^3)$ *cannot be connected with the diffeomorphism* g *by a shortest path in* $\mathcal{D}(M^3)$.

Proof. Rather than the length, we shall estimate an equivalent quantity, the action along the paths.

Definition 7.14. The *action* along a path g_t, $t_1 \leq t \leq t_2$, on the group of diffeomorphisms $\mathcal{D}(M^n)$ of a Riemannian manifold M^n is the quantity

$$\mathrm{j}\{g_t\}_{t_1}^{t_2} = \frac{1}{2} \int_{t_1}^{t_2} \|\dot{g}_t\|_{L^2(M)}^2 \, dt = \frac{1}{2} \int_{t_1}^{t_2} \int_{M^n} \left\| \frac{\partial g_t(x)}{\partial t} \right\|^2 dy \, dt.$$

The action and the length are related via the inequality

(7.2) $$\ell^2 \leq 2\,\mathrm{j}\,(t_2 - t_1).$$

Unlike the length $\ell\{g_t\}_{t_1}^{t_2}$, the action $\mathrm{j}\{g_t\}_{t_1}^{t_2}$ depends on the parametrization, and the equality in (7.2) holds if and only if the parametrization is such that

$$\|\dot{g}_t\|_{L^2(M)}^2 = \int_M \left\| \frac{\partial g_t(x)}{\partial t} \right\|^2 dx \equiv \text{const}.$$

This allows us, in the sequel, to pass freely from one notion to the other.

Suppose there exists a shortest path g_t connecting Id and g; we shall construct another path that has smaller length.

First squeeze the flow g_t by a factor of 2 along the z-direction: Instead of a family of volume-preserving diffeomorphisms of the three-dimensional unit cube M^3, we now have volume-preserving diffeomorphisms of the parallelepiped

$$P_1 = \{(x, z) \in M^3 \mid 0 < z < 1/2\}.$$

Now consider the new (discontinuous) flow $\bar{g}_t$ in the cube $M^3 = P_1 \cup P_2$ that is the above squeezed flow on each of the halves P_1 and P_2; see Fig. 53. (Here P_2 is specified by the condition $1/2 < z < 1$.) It is easy to see that the flow $\bar{g}_t$ is incompressible in M^3 and is, in general, discontinuous on the (invariant) plane $z = 1/2$. Notice also that the flow $\bar{g}_t$ satisfies the same boundary conditions as g_t: $\bar{g}_0 = \text{Id}, \bar{g}_1 = g$.
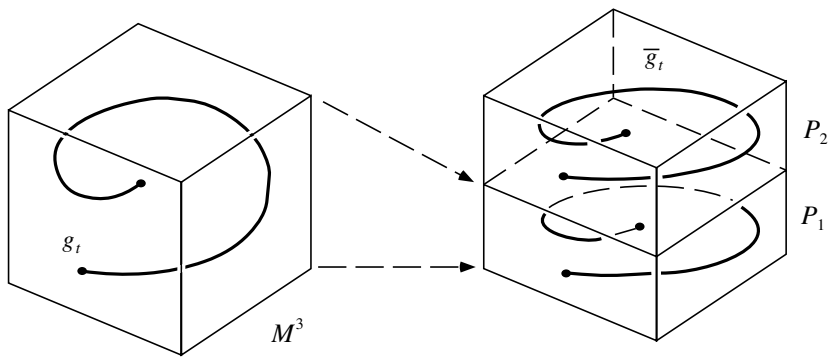
FIGURE 53.   Each of the two parallelepipeds contains the former flow in the cube squeezed by a factor of 2.

Compare the actions along the paths g_t and $\bar{g}_t$. Define the horizontal j_H and vertical j_V components of the action $j = j_H + j_V$ for g_t as follows:

$$j_H\{g_t\}_0^1 = \frac{1}{2}\int_0^1 dt \int_{M^3} \|\frac{\partial x'(x,z,t)}{\partial t}\|^2 \, d^2x \, dz,$$

$$j_V\{g_t\}_0^1 = \frac{1}{2}\int_0^1 dt \int_{M^3} \|\frac{\partial z'(x,z,t)}{\partial t}\|^2 \, d^2x \, dz,$$

and similarly for the path $\bar{g}_t$. (Here $g_t(x,z) = (x',z')$.) From the definition of $\bar{g}_t$, we easily obtain

$$j_H\{\bar{g}_t\}_0^1 = j_H\{g_t\}_0^1, \qquad \text{while} \quad j_V\{\bar{g}_t\}_0^1 = \frac{1}{2}j_V\{g_t\}_0^1.$$

Therefore, the action along the path $\bar{g}_t$ is smaller than that along the path g_t: $j\{\bar{g}_t\}_0^1 < j\{g_t\}_0^1$ if the vertical component of the action is positive, $j_V\{g_t\}_0^1 > 0$.

The last condition, $j_V\{g_t\}_0^1 > 0$, follows from the assumption of the theorem. Indeed, if the vertical component of the action vanishes ($j_V\{g_t\}_0^1 = 0$), then $\partial z'(x,z,t)/\partial t \equiv 0$, and the map $x \mapsto x'(x,z,t)$ for any fixed z and t is an area-preserving diffeomorphism of the square M^2. In this case the action along the flow g_t is

$$j\{g_t\}_0^1 = \int_0^1 dz \left(\int_0^1 dt \int_{M^2} \frac{1}{2}\|\frac{\partial x'(x,z,t)}{\partial t}\|^2 \, d^2x \right),$$

which implies that there is a value $z_0 \in [0,1]$ such that

$$j\{g_t\}_0^1 \geq \int_0^1 dt \int_{M^2} \frac{1}{2}\|\frac{\partial x'(x,z_0,t)}{\partial t}\|^2 \, d^2x.$$

However, this is impossible, since by the assumption, the distance in $\mathcal{D}(M^2)$ from Id to the diffeomorphism $x \mapsto x'(x,z_0,t)\big|_{t=1} = h(x)$ is greater than the distance in $\mathcal{D}(M^3)$ from Id to g, the length of the shortest path g_t in $\mathcal{D}(M^3)$ connecting Id and g.

Hence $j_V\{g_t\}_0^1 > 0$, and we have constructed a discontinuous flow $\bar{g}_t$ (connecting Id and the diffeomorphism g) whose action is less than that of g_t: $j\{\bar{g}_t\}_0^1 < j\{g_t\}_0^1$. Now Theorem 7.12′ follows from the following lemma on a smooth approximation.

Lemma 7.15. *For every $\varepsilon > 0$ there exists a smooth flow $\varphi_t \subset \mathcal{D}(M^3)$, $0 \le t \le 1$, that starts at the identity $\varphi_0 = \text{Id}$, reaches an ε-vicinity of g in the standard L^2-metric, $\text{dist}_{L^2(M^3)}(\varphi_1, g) < \varepsilon$, and whose action approximates the action along the discontinuous path $\bar{g}_t$: $|j\{\bar{g}_t\}_0^1 - j\{\varphi_t\}_0^1| < \varepsilon$.*

To complete the proof, we replace the short but discontinuous path $\bar{g}_t$ by its smooth approximation φ_t. It starts at the identity and ends up at φ_1, L^2-close to g. By Theorem 7.9, there exists a path f_t in $\mathcal{D}(M^3)$, $1 \le t \le 2$, connecting φ_1 and g, and such that

$$\ell\{f_t\}_0^1 \le C\varepsilon^\alpha, \qquad \alpha > 0,$$

and hence the length $\ell\{f_t\}_0^1$ tends to 0 together with ε. It follows that the composite path $\varphi_t \cup f_t$ has length not exceeding $\ell\{\bar{g}_t\} + \varepsilon + C\varepsilon^\alpha$. Finally, observe that for sufficiently small ε, the composite path is shorter than g_t, because $\ell\{\bar{g}_t\}_0^1 < \ell\{g_t\}_0^1$. This completes the proof of Theorem 7.12′ modulo Lemma 7.15. □

Proof of Theorem 7.12. By virtue of Theorem 7.5, for every $C > 0$ there is a diffeomorphism h of the square M^2 such that $\text{dist}_{\mathcal{D}(M^2)}(\text{Id}, h) > C$. On the other hand, if g is a diffeomorphism of the 3-dimensional cube M^3 having the form

$$g(x, z) = (h(x), z), \ x \in M^2, z \in (0, 1),$$

then by Theorem 7.4′, $\text{dist}_{\mathcal{D}(M^3)}(\text{Id}, g) \le 2$. Hence, if $C > 2$, this diffeomorphism g cannot be connected with Id by a shortest path. This completes the proof of Theorem 7.12. □

Proof of Lemma 7.15. For a (discontinuous) flow $\bar{g}_t$, we define (almost everywhere) its Eulerian velocity field by $v(x, t) = \partial \bar{g}_t(\bar{g}_t^{-1}(x))/\partial t$. For a small $\delta > 0$ let M_δ be the set M with the δ-neighborhood of the boundary ∂M removed, and ρ_δ the dilation mappings $M \to M_\delta$. Denote by $v_\delta(x, t) = (\rho_\delta)_* v(x, t)$ the image of the field v under the dilations. By setting $v_\delta \equiv 0$ outside M_δ, we obtain an L^2-vector field in the whole of $\mathbb{R}^3$ that is incompressible in the generalized sense; i.e., the vector field v_δ is L^2-orthogonal to every gradient vector field.

Now define the smooth field

$$w_\delta(x, t) := \int_{\mathbb{R}^3} v_\delta(y, t)\phi_\delta(x - y)dy,$$

as a convolution of the field v_δ with a mollifier $\phi_\delta(x) = \delta^{-3}\phi(x/\delta)$, where $\phi(x) \in C_0^\infty(\mathbb{R}^3)$ and $\int \phi(x)dx = 1$. The field $w_\delta(x, t)$ has compact support $w_\delta(x, t) \in C_0^\infty(M)$ for all t, and as $\delta \to 0$, it converges to the field $v(x, t)$ uniformly on every compact set in M outside ∂M and outside the plane $z = \frac{1}{2}$. Moreover, $w_\delta \to v$ in

$L^2(M)$. This implies that for sufficiently small δ, the smooth flow φ_t obtained by integrating the vector field w_δ satisfies the conditions of Lemma 7.15.   □

7.E. Discrete flows

The proofs of Theorems 7.4, 7.7, and 7.9 are similar and are based on the following discrete approximation of the group $\mathcal{D}(M^n)$ (cf. also [Lax, Mos2]). Split the cube $M^n \subset \mathbb{R}^n$ into N^n identical subcubes. Let M_N be the set of all these cubes. Denote by $\mathcal{D}_N$ the group of all permutations of the set M_N; this is a discrete analogue of the group $\mathcal{D}(M)$ of volume-preserving diffeomorphisms.

Two subcubes $\varkappa, \varkappa' \in M_N$ are called *neighboring* if they have a common $(n-1)$-dimensional face. A permutation $\sigma \in \mathcal{D}_N$ is called *elementary* if each subcube $\varkappa \in M_N$ is either not affected by σ, or $\sigma(\varkappa)$ is a neighbor of $\varkappa$. A sequence of elementary permutations $\sigma_1, \ldots, \sigma_k$ is called a *discrete flow*; the number k is called its *duration*. We say that the discrete flow $\sigma_1, \ldots, \sigma_k$ *connects* the configurations $\sigma, \sigma' \in \mathcal{D}_N$ if $\sigma_k \circ \sigma_{k-1} \circ \cdots \circ \sigma_1 \circ \sigma = \sigma'$.

The following, purely combinatorial, theorem is the cornerstone of the study. It can be regarded as a discrete version of Theorem 7.4.

Theorem 7.16. *For every dimension n there exists a constant $C_n > 0$ such that for every N every two configurations $\sigma, \sigma' \in \mathcal{D}_N$ can be connected by a discrete flow $\sigma_1, \ldots, \sigma_k$ whose duration k is less than $C_n \cdot N$.*

The proof is tedious yet elementary; see [Shn1].

To formulate the discrete analogue of Theorem 7.9, we define the length of a discrete flow (not to be confused with its duration).

Definition 7.17. Let $\sigma_1, \ldots, \sigma_k$ be a discrete flow in M_N. Let each permutation σ_j take m_j subcubes into neighboring ones and leave $N^n - m_j$ subcubes in place. The *length* $\ell\{\sigma_j\}_1^k$ of the discrete flow is (cf. (7.1))

$$\ell\{\sigma_j\}_1^k = \sum_{j=1}^{k} \frac{(m_j/N^n)^{1/2}}{N}.$$

(The reasoning is transparent: A permutation σ_j is approximated by a vector field of magnitude $1/N$ supported on a set of volume m_j/N^n. Then the summands are "L^2-norms" of the permutations.)

The *distance* in $\mathcal{D}_N$ between configurations σ, σ' is defined as

$$\mathrm{dist}_{\mathcal{D}_N}(\sigma, \sigma') = \min \ell\{\sigma_j\}_1^k,$$

where min is taken over all discrete flows (of arbitrary duration) connecting σ and σ'. The L^2-*distance* between $\sigma, \sigma' \in \mathcal{D}_N$ is defined as

$$\mathrm{dist}_{L^2}(\sigma, \sigma') = \left(\sum_{\varkappa \in M_N} \frac{\|\sigma(\varkappa) - \sigma'(\varkappa)\|^2}{N^n} \right)^{\frac{1}{2}},$$

where $\|\sigma(\varkappa) - \sigma'(\varkappa)\|$ is the distance in $\mathbb{R}^3$ between the centers of the corresponding small cubes.

The following is the analogue of Theorem 7.9 on the relation of metrics.

Theorem 7.18. *For every dimension n there are constants $C_n > 0$ and $\alpha_n > 0$ such that for every N and every pair of permutations $\sigma, \sigma' \in \mathcal{D}_N$,*

$$\operatorname{dist}_{\mathcal{D}_N}(\sigma, \sigma') \leq C_n(\operatorname{dist}_{L^2}(\sigma, \sigma'))^{\alpha_n}.$$

The proof is an inductive multistep construction of a "short" discrete flow connecting two given L^2-close discrete configurations; see [Shn1]. The explicit construction in the three-dimensional case ($n = 3$) yields $\alpha_3 \geq 1/64$.

7.F. Outline of the proofs

The proof of Theorems 7.4 and 7.9 proceeds as follows.

Let $g \in \mathcal{D}(M^n)$ with $n \geq 3$. We construct a path $g_t \subset \mathcal{D}(M^n)$, $0 \leq t \leq 1$, that connects the identity and g ($g_0 = \operatorname{Id}$, $g_1 = g$), and such that

$$\ell\{g_t\}_0^1 \leq C(\operatorname{dist}_{L^2}(\operatorname{Id}, g))^\alpha.$$

First of all, we prove that g can be approximated by some permutation $\sigma \in \mathcal{D}_N$ of small cubes for sufficiently large N. Furthermore, for every $\varepsilon > 0$ there exists a discontinuous, piecewise-smooth flow ξ_τ, $0 \leq \tau \leq 1$, connecting σ and g, and such that $L\{\xi_\tau\}_0^1 < \varepsilon$. (More precisely, for every τ the mapping $\xi_\tau : M^n \to M^n$ is smooth in every small cube $\varkappa \in M_N$, discontinuous on interfaces between neighboring cubes $\varkappa$, and measure-preserving.)

Construct the "short" discrete flow $\sigma_1, \ldots, \sigma_k$ connecting Id and σ in $\mathcal{D}_N$ and satisfying the conclusion of Theorem 7.18. One can show that there exists a discontinuous flow η_t, $0 \leq t \leq 1$, that interpolates the flow $\sigma_1, \ldots, \sigma_k$ at the moments $t = j/k$ for all $j = 1, \ldots, k$ ($\eta_{t=j/k} = \sigma_j \circ \sigma_{j-1} \circ \cdots \circ \sigma_1$) and has the same order of length:

$$\ell\{\eta_t\}_0^1 \leq \operatorname{const} \cdot \ell\{\sigma_j\}_1^k.$$

Therefore, the composition of the paths η_t and ξ_t is a discontinuous flow connecting Id and g and having a controllable length.

The final step is the smoothening of the latter flow provided by the following

Lemma 7.19. *Let $g \in \mathcal{D}(M^n)$, and let $\zeta_t : M^n \to M^n$ be a discontinuous measure-preserving flow such that $\zeta_0 = \operatorname{Id}$, $\zeta_1 = g$, and ζ_t is smooth in every small cube $\varkappa \in M_N$. If $n \geq 3$, then for every $\varepsilon > 0$ there exists a smooth flow $g_t : M^n \to M^n$, $0 \leq t \leq 1$, with the same boundary conditions $g_0 = \operatorname{Id}$, $g_1 = g$ as ζ_t, and such that $\ell\{g_t\}_0^1 < \ell\{\zeta_t\}_0^1 + \varepsilon$.*

The flow g_t coincides with ζ_t in every cube $\varkappa_\delta = \{x \in \varkappa \mid \operatorname{dist}_{\mathbb{R}^n}(x, \partial\varkappa) > \delta\}$ for some small δ. The most subtle point in extending the flow is to define it on the

δ-neighborhood of all subcube faces the "froth-like" domain $K_\delta := \bigcup_{\varkappa \in M_N} (\varkappa \setminus \varkappa_\delta)$. Here we use the fact that the fundamental group of the domain K_δ is trivial $(\pi_1(K_\delta) = 0)$, which is true if $n \geq 3$. Theorem 7.9 (and Theorem 7.4 as a particular case) follows; see [Shn1] and analogous arguments in the proof of Theorem III.3.3 in Section III.3 for the details.

The proof of Theorem 7.7 is similar, but more complicated; we refer to [Shn2].

7.G. *Generalized flows*

We now return to the problem of finding the shortest paths in the group of volume-preserving diffeomorphisms $\mathcal{D}(M^n)$. We already know that there exist pairs of diffeomorphisms that cannot be connected by a smooth flow of minimal length. Is there, however, some wider class of flows (say, discontinuous or measurable) where the minimum is always attainable? This problem has been resolved by Y. Brenier [Bre1]. He found a natural class of "generalized incompressible flows" for which the variational problem is always solvable.

The generalized flows (GF) are a far-reaching generalization of the classical flows, where fluid particles are not only allowed to move independently of each other, but also their trajectories may meet each other: The particles may split and collide. The only restrictions are that the density of particles remains constant all the time and that the mean kinetic energy is finite. The formal definition of the GF is presented below.

Let $X = C([0, 1]; M^n)$ be the space of all parametrized continuous paths $x(t)$ in M^n. Fix a diffeomorphism $g \in \mathcal{D}(M^n)$.

Definition 7.20. *A generalized flow (GF) in M^n connecting the diffeomorphisms* Id *and* g *is a probabilistic measure* $\mu\{dx\}$ *in the space X satisfying the following conditions:*

(i) *For every Lebesgue-measurable set $A \subset M^n$, and every $t_0 \in [0, 1]$,*

$$\mu\{x(t) \mid x(t_0) \in A\} = \text{mes } A$$

(this may be called incompressibility).

(ii) *For μ-almost all paths $x(t)$, the action along each of them is finite*

$$j\{x(\cdot)\} = \frac{1}{2} \int_0^1 \|\frac{\partial x(t)}{\partial t}\|^2 dt < \infty,$$

and so is the "total action"

$$j\{\mu\} = \int_X j\{x(\cdot)\}\mu\{dx\} < \infty$$

(finiteness of action).

(iii) *For μ-almost all paths $x(t)$, the endpoints $x(0)$ and $x(1)$ are related by means of the diffeomorphism g: $x(1) = g(x(0))$ (boundary condition).*

Thus, a generalized flow $\mu\{dx\}$ can be thought of as a random process. In general this process is neither Markov nor stationary. This notion is very similar to the notion of a polymorphism, appearing in the work of Neretin [Ner2]. Polymorphisms arise as a natural domain for the extensions of representations of diffeomorphism groups.

Every smooth flow $g_t \subset \mathcal{D}(M)$ may be regarded as a generalized flow if we associate to g_t the measure $\mu_{(g_t)}\{dx\}$ such that for every measurable set $Y \subset X$ its $\mu_{(g_t)}$-measure is equal to the measure of points whose trajectories belong to Y:

$$\mu_{(g_t)}(Y) = \text{mes}\{a \in M^n \big| \{g_t(a)\} \in Y\}.$$

This measure is concentrated on the n-dimensional set of trajectories $g_t(a)$ of the flow g_t.

Another example of a GF is a *multiflow*, that is, a convex combination of GFs, corresponding to smooth flows. In other words, in the multiflows different portions of fluid move (penetrating each other; see Fig. 54) in different directions within the same volume! Generic GFs are much more complicated than multiflows.

Theorem 7.21 [Bre1]**.** *For every diffeomorphism* $g \in \mathcal{D}(M^n)$ *there always exists a generalized flow* μ *(with the boundary conditions* Id *and* g*) that realizes the minimum of action,*

$$j\{\mu\} = \min_{\mu'} j\{\mu'\},$$

where the minimum is taken over generalized flows μ' *connecting the identity* Id *and the diffeomorphism* g*.*

Thus, although in our example the infimum cannot be assumed among smooth flows, there exists a generalized flow minimizing the action (as well as the length); see also [Roe].

In fact, Theorem 7.21 is even more general, since it applies equally to discontinuous and orientation-changing maps g. In the latter case the minimizing GF is especially interesting because the fluid is being turned "inside out"! No measurable flow, or even multiflow, can produce such a transformation. These problems are nontrivial even in the one-dimensional case.

Here are two beautiful examples found by Brenier. Let $g_1(x)$ and $g_2(x)$ be the transformations of the segment $[0, 1]$, defined, respectively, as a flip-flop map $g_1(x) := 1 - x$, $x \in [0, 1]$, and as an interval-exchange map $g_2 \colon [0, 1/2] \leftrightarrow [1/2, 1]$.

Figures 54a and 54b display the trajectories of fluid particles for the minimal flows connecting Id with g_1 and g_2, respectively. In the first case, each fluid particle splits at $t = 0$ into a continuum of trajectories of "smaller" particles; they move independently, pass through all the points of the segment, and then coalesce at $t = 1$. In the second case, the GF is a multiflow (more precisely, a 2-flow). For more examples of exotic minimal GFs, see [Bre1, 2, 3].
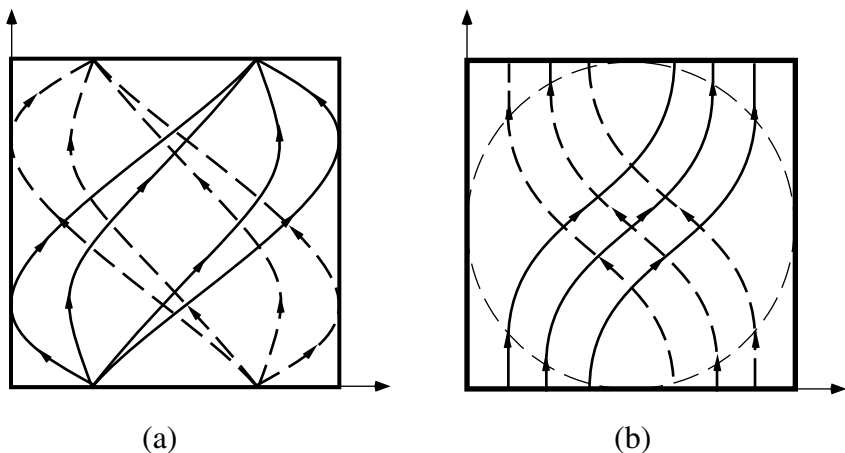
FIGURE 54. Trajectories of particles in the GF, corresponding to (a) the flip of the interval [0, 1] and (b) the interval-exchange map.

An important question is to what extent these minimal GFs may be regarded as generalized solutions of the Euler equation. The similarity between these generalized flows and the "true" solutions extends very far: For example, for every GF μ there exists a scalar function $p(x, t)$, playing the role of pressure [Bre2], such that for almost every fluid particle its acceleration at almost every (x, t) is equal to $-\nabla_x p$! The minimal GFs are generalized solutions of the mass transport problem (of the so-called Kantorovich problem). However, their hydrodynamical meaning is not yet completely understood.

7.H. Approximation of fluid flows by generalized ones

Generalized flows have proved to be a powerful and flexible tool for studying the structure of the space $\mathcal{D}(M^n)$ of volume-preserving diffeomorphisms. The key role here is played by the following approximation theorem.

Theorem 7.22 [Shn5]. *Let $g \in \mathcal{D}(M^n)$, $n \geq 3$, be a smooth volume-preserving diffeomorphism. Then each generalized flow $\mu\{dx\}$ connecting* Id *and g can be approximated (together with the action) by smooth incompressible flows: There exists a sequence of smooth incompressible flows $g_t^{(k)}$ connecting* Id *and g such that as $k \to \infty$,*

(*i*) *the measures $\mu_{g_t^{(k)}}$ weak*-converge in X to the measure μ;*

(*ii*) *the actions along $g_t^{(k)}$ converge to the total action along $\mu\{dx\}$:*

$$j\{g_t^{(k)}\}_0^1 \to j\{\mu\}_0^1.$$

Here weak*-convergence means that for every bounded continuous functional $\varphi\{x(\cdot)\}$ on X,

$$\langle \varphi\{x\}, \mu_{g_t^{(k)}}\{dx\}\rangle \to \langle \varphi(x), \mu\{dx\}\rangle, \qquad \text{as} \quad k \to \infty.$$

We shall not present here the (lengthy) proof of this theorem, referring to [Shn5] instead. An immediate consequence of this theorem and formula (7.2) is the following estimate on the distances in $\mathcal{D}(M^n)$, $n \geq 3$.

Corollary 7.23. *If $n \geq 3$, then for every diffeomorphism $g \in \mathcal{D}(M^n)$,*

$$\text{dist}(\text{Id}, g) = \inf (2 \cdot \text{j}\{\mu\}_0^1)^{1/2},$$

where the infimum is taken over all generalized flows μ connecting Id *and g.*

Thus, to estimate the distance between Id and $g \in \mathcal{D}(M^n)$, we may try to construct a GF connecting Id and g and having the smallest possible action. Then Lemma 7.23 guarantees a majorant for the distance.

Example 7.24 (= Theorem 7.4). Let us estimate the diameter diam $\mathcal{D}(M^n)$ of the group of volume-preserving diffeomorphisms of the n-dimensional unit cube. An accurate computation of all the intermediate constants in the proof of Theorem 7.4 for $n = 3$ yields diam $\mathcal{D}(M^3) < 100$, which is very far from reality. Here we prove

Theorem 7.4′ [Shn5]. *If the dimension $n \geq 3$, then* diam $\mathcal{D}(M^n) \leq 2\sqrt{n/3}$.

Proof. We use a construction close to that of Y. Brenier, who proved that all fluid configurations on the torus are attainable by GFs [Bre1].

The required GF is constructed as follows. At $t = 0$ every fluid particle in the cube splits into a continuum of particles moving in all directions. Having originated at a point y, this "cloud" (of cubical form) expands, and at $t = 1/2$ it fills out the whole cube M^n with a constant density. During the second half of the motion ($1/2 \leq t \leq 1$) the "cloud" shrinks and collapses at $t = 1$ to the point $g(y)$. All "clouds" expand and shrink simultaneously, and the overall density remains constant for all t.

More accurately, suppose that the cube M^n is given by the inequalities $|x_i| < \frac{1}{2}$, $i = 1, \ldots, n$. Let Γ be a discrete group of motions generated by the reflections in the faces of M^n. For each initial point $y \in M^n$ and a velocity vector $v \in M^n$, we define the corresponding billiard trajectory in M^n for the time $0 \leq t \leq 1/2$, i.e., the path $x_{y,v}(t) \subset M$, where

$$x_{y,v}(t) := \Gamma(y + 4vt) \cap M^n.$$

Given a point y, the end-point mapping $\phi_y : v \to x_{y,v}(1/2)$ is a 2^n-fold covering of M^n, and moreover, ϕ_y is volume-preserving. These billiard trajectories are

trajectories of the "microparticles" into which every initial point y splits. At $t = \frac{1}{2}$ the microparticles fill M^n uniformly, and after this moment they move along other billiard trajectories, gathering at the point $g(y)$ at the end. All particles split and move independently in the same manner; incompressibility is fulfilled automatically.

Let M_y, M_v, M_z, M_u be 4 copies of the cube M^n with coordinates y, v, z, u, respectively. Define a set $\Omega \subset M_y \times M_v \times M_z \times M_u$ that consists of all four-tuples $\omega = (y, v, z, u) \in \Omega$ such that $z = g(y)$ and such that the endpoints of the corresponding trajectories coincide: $x_{y,v}(1/2) = x_{z,u}(1/2)$.

Denote by $d\omega = 2^{-n} dy \, dv$ the normed volume element on Ω. Then the required GF μ is the following random process in M^n with probability space $(\Omega, d\omega)$:

$$x(t, \omega) = \begin{cases} x_{y,v}(4t), & 0 \leq t \leq 1/2, \\ x_{g(y),u}(4 - 4t), & 1/2 \leq t \leq 1, \end{cases}$$

where $\omega = (y, v, g(y), u) \in \Omega$. The action of this GF is

$$j\{\mu\}_0^1 = \frac{1}{2} \int_{M_v} 16v^2 \, dv = \frac{n}{2} \int\limits_{-1/2}^{1/2} 16x^2 \, dx = \frac{2n}{3}.$$

By virtue of Corollary 7.23 this implies that the distance between the identity Id and the diffeomorphism g (which has been chosen arbitrarily) is majorated as follows:

$$\mathrm{dist}(\mathrm{Id}, g) \leq \sqrt{2 \cdot j\{\mu\}_0^1} = 2\sqrt{n/3}.$$

Hence, the diameter of the group $\mathcal{D}(M^n)$ has the same upper bound.   □

Analogous (though much longer) reasoning proves Theorem 7.9′, which minorates the Hölder exponent α_n, $n \geq 3$, for the embedding of $\mathcal{D}(M^n)$ into $L^2(M, \mathbb{R}^n) : \alpha_n \geq 2/(n+4)$; see [Shn5]. Given g, we construct explicitly the GF satisfying that estimate. Our constructions are possibly not optimal, and a natural question is to find the best possible estimate for the diameter and for the exponent α_n. Is the latter equal to or less than 1? Both possibilities are interesting.

7.I. Existence of cut and conjugate points on diffeomorphism groups

One more application of the techniques of generalized flows is the proof of the existence of cut points on the space $\mathcal{D}(M^n)$; cf. Section 6.

Definition 7.25. Let $g_t \subset \mathcal{D}(M^n)$ be a geodesic trajectory on the group of diffeomorphisms. We call a point g_{t_c} on the trajectory the *first cut* of the initial point g_0 along g_t if the geodesic g_t has minimal length among all curves connecting g_0 and g_τ for all $\tau < t_c$, and it ceases to minimize the length as soon as $\tau > t_c$ (i.e., for every $\tau > t_c$ there exists a curve g_t' connecting g_0 and g_τ whose length is less than

the length of the segment $\{g_t \mid 0 < t < \tau\}$). We call a point g_{t_c} the *first local cut* if it is the first cut, and the curve g_t' may be chosen arbitrarily close to the geodesic segment $\{g_t \mid 0 < t < \tau\}$ for every $\tau > t_c$.

On a complete finite-dimensional Riemannian manifold, the cut point of a point g_0 comes no later than the first conjugate point of g_0. The example of a flat torus shows that one can have cut points but no conjugate points. But in the finite-dimensional case, the first local cut point is always a conjugate point; so, all cut points on the torus are nonlocal, which is evident. In the case of diffeomorphism groups the precise relationship between cut points and conjugate points has yet to be clarified.

In the two-dimensional case ($n = 2$) the conjugate points on certain geodesics in $\mathcal{D}(T^2)$ were found by G. Misiołek [Mis2]; see Section 6. It is curious that for $n \geq 3$ there are local cut (and, probably, conjugate?) points on *every* sufficiently long geodesic curve. This is a consequence of the following result.

Theorem 7.26 [Shn5]. *Let $\{g_t \mid 0 \leq t \leq T\} \subset \mathcal{D}(M^n)$ be an arbitrary path on the group and $n \geq 3$. If the length of the path exceeds the diameter of the group, $\ell\{g_t\}_0^T > \operatorname{diam} \mathcal{D}(M^n)$, then there exists a path $\{g_t' \mid 0 \leq t \leq T\} \subset \mathcal{D}(M^n)$ with the same endpoints g_0 and g_T that*

- (i) *is uniformly close to g_t (i.e., for every $\varepsilon > 0$ there exists a path g_t' such that $\operatorname{dist}_{\mathcal{D}(M)}(g_t, g_t') < \varepsilon$ for every $t \in [0, T]$) and*
- (ii) *has a smaller length: $\ell\{g_t'\}_0^T < \ell\{g_t\}_0^T$.*

In other words, if the geodesic segment g_t is long enough ($\ell\{g_t\}_0^T > \operatorname{diam} \mathcal{D}(M^n)$), then there exists a local cut point g_{t_c} with $t_c < T$. A shorter path can be chosen arbitrarily close to the initial geodesic, which on a complete finite-dimensional manifold would imply the existence of a conjugate point.

Proof. Let h_t, $0 \leq t \leq T$, be a path in $\mathcal{D}(M^n)$ connecting g_0 and g_T and such that

$$\ell\{h_t\}_0^T < \operatorname{diam} \mathcal{D}(M^n) + \frac{\delta}{2} < \ell\{g_t\}_0^T$$

for some small $\delta > 0$. Such a path exists by the definition of diameter.

Assume that the parametrization of the paths g_t and h_t is chosen in such a manner that $\|\dot{g}_t\| = \operatorname{const}$, $\|\dot{h}_t\| = \operatorname{const}$, and hence we have the inequality $\operatorname{j}\{h_t\}_0^T < \operatorname{j}\{g_t\}_0^T$ for the actions as well.

Let μ_{g_t}, μ_{h_t} be the GFs corresponding to the classical flows g_t, h_t. Consider the convex combination $\bar{\mu}$ of the measures μ_{g_t}, μ_{h_t} in the space X: $\bar{\mu} := (1 - \lambda)\mu_{g_t} + \lambda\mu_{h_t}$ for some $0 < \lambda < 1$. Then the total action for the generalized flow $\bar{\mu}$ is

$$\operatorname{j}\{\bar{\mu}\}_0^T = (1 - \lambda)\operatorname{j}\{g_t\}_0^T + \lambda\operatorname{j}\{h_t\}_0^T < \operatorname{j}\{g_t\}_0^T.$$

To return to the classical flows we use approximation Theorem 7.22. It guarantees that there exists a smooth flow f_t, $0 \leq t \leq T$, connecting g_0 and g_T and

weakly*-approximating the GF μ together with its action, so that $j\{f_t\}_0^T < j\{g_t\}_0^T$. The flow f_t, certainly, depends on λ, and it is easy to see that for small λ the flow f_t is L^2-close to g_t:

$$\mathrm{dist}_{L^2}(f_t, g_t) < C \cdot \lambda^{1/2}.$$

Hence, for sufficiently small λ, these two flows are close on the group by virtue of Theorem 7.9: $\mathrm{dist}_{\mathcal{D}(M^n)}(f_t, g_t) < \varepsilon$ for all $0 \leq t \leq T$. This completes the proof of Theorem 7.26. $\square$

For other applications of the generalized flows and for more detail we refer to [Shn5, Shn8].

§8. Infinite diameter of the group of Hamiltonian diffeomorphisms and symplecto-hydrodynamics

The picture changes drastically when we turn from the group of volume-preserving diffeomorphisms of three- (and higher-) dimensional manifolds to area-preserving diffeomorphisms of surfaces. Practically none of the aspects under consideration in the preceding section (such as metric properties and diameter of the group, existence of solutions for the variational problem of Cauchy and Dirichlet types, or completion of the group and description of attainable diffeomorphisms) can be literally transferred to this case. It is natural to describe the properties of the groups of area-preserving diffeomorphisms of surfaces in the more general setting of diffeomorphisms of arbitrary symplectic manifolds.

Definition 8.1. A *symplectic manifold* (M, ω) is an even-dimensional manifold M^{2n} endowed with a nondegenerate closed differential two-form ω.

A *group of symplectomorphisms* consists of all diffeomorphisms $g : M \to M$ that preserve the two-form ω (i.e., $g^*\omega = \omega$). We will be considering symplectomorphisms belonging to the identity connected component in the symplectomorphism group, and particularly those symplectomorphisms that can be obtained as the time-one map of a Hamiltonian flow. By the Hamiltonian flow we mean the flow of a time-dependent Hamiltonian vector field (having a single-valued Hamiltonian function). Such symplectic diffeomorphisms of M are called *Hamiltonian*. Denote the group of Hamiltonian diffeomorphisms by $\mathrm{Ham}(M)$ and the corresponding Lie algebra of Hamiltonian vector fields by $\mathrm{ham}(M)$.

For a two-dimensional manifold the symplectic two-form ω is an area form, and the group of area- and mass-center-preserving diffeomorphisms $S\,\mathrm{Diff}_0(M^2)$ coincides with the group $\mathrm{Ham}(M)$. The role of the group $\mathrm{Ham}(M)$ in plasma dynamics is similar to that of the group $S\,\mathrm{Diff}(M)$ in ideal fluid dynamics.

The study of geodesics of right-invariant metrics on symplectomorphism groups is an interesting and almost unexplored domain. It might be called symplecto-

hydrodynamics, and it is a rather natural generalization of two-dimensional hydrodynamics. The relation becomes even more transparent for complex or almost complex manifolds, where the metric $\langle \, , \, \rangle$ is related to the symplectic structure ω by means of the relation $\langle \xi, \eta \rangle = \omega(\xi, i\,\eta)$.

The symplecto-hydrodynamics in higher dimensions differs drastically from that in dimension two. For instance, every bounded domain on the plane can be embedded in any other domain of larger area by a symplectomorphism (i.e., by a diffeomorphism preserving the areas). Already in dimension four this is not always the case: Even some ellipsoids in a symplectic space cannot be embedded in a ball of larger volume by a symplectomorphism [Gro]. For example, the ellipsoid $\frac{1}{a^2}(p_1^2 + q_1^2) + \frac{1}{b^2}(p_2^2 + q_2^2) \leq 1$ cannot be sent into a ball $p_1^2 + q_1^2 + p_2^2 + q_2^2 \leq R^2$ of bigger volume if $R < \max(a, b)$. Moreover, a "symplectic camel" (a bounded domain in the symplectic four-dimensional space) cannot go through the eye of a needle (a small hole in the three-dimensional wall), while in volume-preserving hydrodynamics such a percolation through an arbitrarily small hole is always possible in any dimension.

Thus the preservation of the symplectic structure ω of the phase space M by the Hamiltonian phase flow implies some peculiar restrictions on the resulting diffeomorphisms, making symplectomorphisms scarce among the volume-preserving maps in dimensions ≥ 4. (Moreover, the group of symplectomorphisms of a symplectic manifold is C^0-closed in the group of all diffeomorphisms of the manifold; i.e., in general, a volume-preserving diffeomorphism cannot be approximated by symplectic ones [El1, Gro]). These restrictions might even imply some unexpected phenomena in statistical mechanics, where, in spite of the symplectic nature of the problem, one usually takes into account the first integrals and volume preservation only and freely permutes the particles of the phase space. One may also hope that symplecto- (contacto-, conformo-) hydrodynamics will find other physically interesting applications. In this section we will describe a few results known in symplecto-hydrodynamics.

We will concentrate mostly on two main metrics with which the group $\mathrm{Ham}(M)$ can be equipped. The first one is the right-invariant metric, which arises from the kinetic energy and is responsible for hydrodynamic applications (we follow [ER1, 2]). The second one is the bi-invariant metric introduced in [Hof] (and studied in [E-P, LaM]), which has turned out to be a powerful tool in symplectic geometry and topology.

8.A. *Right-invariant metrics on symplectomorphisms*

Let (M^{2n}, ω) be a compact *exact* symplectic manifold. This means that the symplectic form ω is a differential of a 1-form θ: $\omega = d\theta$.

Such a manifold neccessarily has a nonempty boundary. Otherwise the integral

$$\int_M \omega^n = \int d(\theta \wedge \omega^{n-1})$$

would vanish, which is impossible since the $2n$-form $\mu = \omega^n$ is a volume form on M. We fix a Riemannian metric on M with the same volume element.

Definition 8.2. The right-invariant L^p-*metric* on Ham(M) is determined by the L^p-norm ($p \geq 1$) on Hamiltonian vector fields ham(M) at the identity of the group (for hydrodynamics, one needs the L^2-case corresponding to the kinetic energy of a fluid). Given a path $\{g_t \mid t \in [0, 1]\} \subset$ Ham(M), we define its L^p-*length* $\ell_p(\{g_t\})$ by the formula

$$\ell_p(\{g_t\}) = \int_0^1 \|\frac{dg_t}{dt}\|_{L^p} \, dt = \int_0^1 \left(\int_M \|\frac{dg_t}{dt}\|^p \mu \right)^{1/p} dt.$$

The length functional ℓ_p gives rise to the *distance function* dist$_p$ on Ham(M) by

$$\text{dist}_p(f, g) = \inf \ell_p(\{g_t\}),$$

where the infimum is taken over all paths g_t joining $g_0 = f$ and $g_1 = g$. Finally, define the *diameter* of the group by

$$\text{diam}_p(\text{Ham}(M)) := \sup_{f, g \in \text{Ham}(M)} \text{dist}_p(f, g).$$

Theorem 8.3 (= 7.5″) [ER2]. *The diameter* diam$_p$(Ham(M)) *of the group of Hamiltonian diffeomorphisms* Ham(M) *is infinite in any right-invariant L^p-metric.*

Note that the strongest result is that for the L^1-norm, since

$$\ell_p(*) \geq C(M, p) \cdot \ell_1(*).$$

Remark 8.4. Contrary to the volume-preserving case (for dim $M \geq 3$), the infiniteness of the diameter of the symplectomorphism group has a local nature, and it is not related to the topology of the underlying manifold. The source of the distinction between these two cases is in the different topologies of the corresponding groups of linear transformations. The fundamental group of the group of linear symplectic transformations Ham$(2n)$ is infinite, while it is finite in the volume-preserving case of $SL(2n)$ for $n > 1$.

To give an example of a "long path" in a group of Hamiltonian diffeomorphisms, we consider the unit disk $B^2 \subset \mathbb{R}^2$ with the standard volume form. Then such a path on Ham(B^2) is given, for instance, by the Hamiltonian flow with Hamiltonian function $H(x, y) = (x^2 + y^2 - 1)^2$ for a long enough period of time (Fig. 55). The final symplectomorphism is sufficiently far away from the identity diffeomorphism Id $\in$ Ham(B^2), since two-dimensionality prevents "highly twisted clusters of particles" to untwist via a short path.

This allows one to present the following example of an unattainable diffeomorphism of the square [Shn2]. It corresponds to the time-one map of the flow whose Hamiltonian function is depicted in Fig. 56. It has "hills" of infinitely increasing
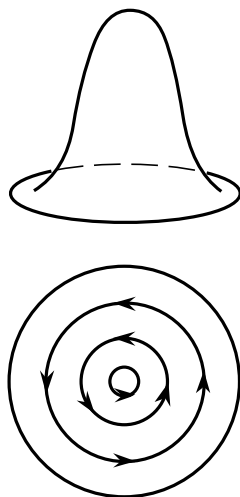
FIGURE 55.   Profile of the Hamiltonian function and the trajectories of the corresponding flow, which is "a long path" on the group $\mathrm{Ham}(B^2)$.

height and with supports on a sequence of disks convergent to the boundary of the square.
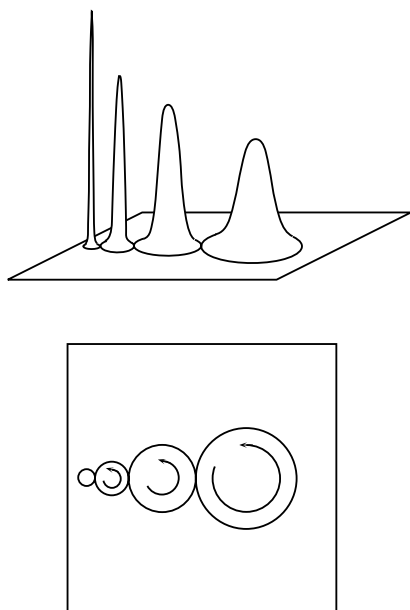
FIGURE 56.   An unattainable diffeomorphism of the square.

We will prove Theorem 8.3 for the case of the L^2-norm and the group of symplectomorphisms of the ball B^{2n} that are fixed on the boundary ∂B [ER1]. The main ingredient of the proof is the notion of the Calabi invariant.

8.B. Calabi invariant

Consider the group $\mathrm{Ham}_\partial(B)$ of the Hamiltonian diffeomorphisms of the ball (B^{2n}, ω) stationary on the sphere ∂B (ω being the differential of a 1-form θ, say, the standard symplectic structure $\omega = \sum dp_i \wedge dq_i$ in $\mathbb{R}^{2n}$).

Proposition 8.5. *Given a 1-form θ on the ball B and a Hamiltonian diffeomorphism $g \in \mathrm{Ham}_\partial(B)$ fixed on the sphere ∂B, there exists a unique function $h : B^{2n} \to \mathbb{R}$ vanishing together with its gradient on ∂B and such that*

$$(8.1) \qquad\qquad \theta - g^*\theta = dh.$$

Proof. The 1-form $\theta - g^*\theta$ is closed ($d(\theta - g^*\theta) = d\theta - g^*d\theta = \omega - g^*\omega = 0$) and hence exact in the ball B^{2n}. Therefore, it is the differential of some function h. The vanishing property for h is provided by the condition that g is steady on the boundary. $\square$

Lemma–definition 8.6. *The integral of the function h over the ball B does not depend on the choice of the 1-form θ satisfying $d\theta = \omega$. The* Calabi invariant *of the Hamiltonian diffeomorphism g is this integral divided by $(n+1)$:*

$$\mathrm{Cal}(g) := \frac{1}{n+1} \int_B h\,\omega^n.$$

Proof. The form θ is defined modulo the differential of a function. Under the change $\theta \mapsto \tilde{\theta} = \theta + df$, the function h becomes $\tilde{h} = h + (f - g^*f)$, since the differential commutes with pullbacks. The forms $(g^*f)\omega^n$ and $f\omega^n$ have the same integrals, since the map g preserves the symplectic structure ω. Then the integral of h is preserved:

$$\int_B \tilde{h}\,\omega^n = \int_B h\,\omega^n + \int_B (f - g^*f)\,\omega^n = \int_B h\,\omega^n = \mathrm{Cal}(g).$$

$\square$

Lemma–definition 8.6 also holds for an arbitrary symplectomorphism g of the ball, fixed on the boundary. However, for Hamiltonian diffeomorphisms, there is the following alternative description of the Calabi invariant.

Let $\mathrm{ham}_\partial(B)$ be the Lie algebra of the group $\mathrm{Ham}_\partial(B)$ of Hamiltonian diffeomorphisms of the ball. It consists of the Hamiltonian vector fields vanishing on the boundary sphere ∂B. We shall identify it with the space of Hamiltonian functions

H, normalized by the condition that H and its differential both vanish on ∂B. (Notice that the definition of the Hamiltonian function corresponding to a vector field from $\mathrm{ham}_\partial(B)$ is the infinitesimal version of relation (8.1).)

Let $g \in \mathrm{Ham}_\partial(B)$ be a Hamiltonian diffeomorphism of the n-dimensional ball B. Consider any path $\{g_t \mid 0 \le t \le T, \, g(0) = \mathrm{Id}, \, g(T) = g\}$ on the group $\mathrm{Ham}_\partial(B)$ connecting the identity element with g. The path may be regarded as the flow of a time-dependent Hamiltonian vector field on B whose normalized Hamiltonian function H_t (defined on $B^{2n} \times [0, T]$) vanishes on ∂B along with its differential.

Theorem 8.7 [Ca]. *The integral of the Hamiltonian function H_t over $B^{2n} \times [0, T]$ is equal to the Calabi invariant of the symplectomorphism g:*

$$(8.2) \qquad \mathrm{Cal}(g) = \int_0^T \left(\int_B H_t \, \omega^n \right) dt.$$

In particular, this integral does not depend on the connecting path, that is, on the choice of time-dependent Hamiltonian H_t, provided that the time-one map $g(T) = g$ is fixed.

Geometrically, the Calabi invariant is the volume in the $(2n + 2)$-dimensional space $\{(x, t, z)\} = B^{2n} \times [0, T] \times \mathbb{R}$ under the graph of the function $(x, t) \mapsto z = H_t(x)$.

Lemma 8.8. *The Calabi invariant* $\mathrm{Cal} \colon \mathrm{Ham}_\partial(B) \to \mathbb{R}$ *is the group homomorphism of the group of Hamiltonian diffeomorphisms of B (fixed on the boundary) onto the real line.*

Proof of Lemma. Let $g_1, g_2 \in \mathrm{Ham}_\partial(B) \to \mathbb{R}$ be two Hamiltonian diffeomorphisms, and $g = g_2 \circ g_1$. We have to show that the corresponding functions h and h_i, $i = 1, 2$, vanishing on the boundary ∂B and determined by the condition (8.11), satisfy the relation

$$\int h \, \omega^n = \int h_1 \, \omega^n + \int h_2 \, \omega^n.$$

The latter holds, since

$$dh = \theta - g^*\theta = \theta - g_2^*\theta + g_2^*\theta - (g_2 \circ g_1)^*\theta = dh_2 + g_2^*(dh_1),$$

and because g_2 preserves the symplectic form ω. $\qquad \square$

Remark 8.9. The kernel of the Calabi homomorphism (formed by the Hamiltonian diffeomorphisms whose Calabi invariant vanishes) is a simple group, the commutant of $\mathrm{Ham}_\partial(B)$; see [Ban]. (The *commutant of a group* consists of the products of commutators of the group elements.)

The fact that the Hamiltonian diffeomorphisms whose Calabi invariant vanishes form a connected normal subgroup is evident (one can multiply the time-dependent Hamiltonian by a constant). The Lie algebra of this subgroup consists of the Hamiltonian vector fields whose normalized Hamiltonian functions has zero integral. The fact that the subgroup consists of the products of commutators in $\mathrm{Ham}_\partial(B)$ is similar to the following. The Hamiltonian functions with vanishing integral are representable as finite sums of Poisson brackets of functions from $\mathrm{ham}_\partial(B)$.

The subgroup of Hamiltonian diffeomorphisms with vanishing Calabi invariant has an infinite diameter, just as the ambient group of all Hamiltonian diffeomorphisms of the ball [ER2].

More generally, the Calabi invariant is the homomorphism of the group $\mathrm{Symp}_\partial(B)$ of all symplectomorphisms of the ball (fixed on the boundary) to $\mathbb{R}$. We do not know whether the group of symplectomorphisms $\mathrm{Symp}_\partial(B)$ of a $2n$-dimensional ball (fixed on the boundary) and this normal subgroup $\{\mathrm{Cal}(g) = 0\}$ are simply connected. The group $\mathrm{Symp}_\partial(B)$ is known to be contractible for $n = 1$ [Mos1] and for $n = 2$ [Gro].

Proof of Theorem 8.7. Let $\{g_t \in \mathrm{Ham}_\partial(B) \mid 0 \leq t \leq T, g(0) = \mathrm{Id}, g(T) = g\}$ be a path of Hamiltonian diffeomorphisms with the time-dependent Hamiltonian function H_t.

Owing to the homomorphism property of Cal, it is enough to prove the relation

$$\int_B h\,\omega^n = (n+1) \int_0^T \left(\int_B H_t\,\omega^n \right) dt$$

for an "infinitesimally short" period of time $[0, T]$. In other words, we differentiate this relation in t at $t = 0$, and will prove the identity

$$\int_B \left(\frac{d}{dt} h \right) \omega^n = (n+1) \int_B H_0\,\omega^n.$$

Note that the time derivative at $t = 0$ of the left-hand side of formula (8.1) for the diffeomorphism g_t is, by definition, minus the Lie derivative of the 1-form θ along the Hamiltonian vector field $v = \frac{d}{dt}|_{t=0} g_t$ generated by the function $H = H_0$:

(8.3)
$$-L_v\theta = d\left(\frac{d}{dt} h \right).$$

We apply the homotopy formula $L_v = i_v d + d i_v$ (see Section I.7.B) to the Lie derivative $L_v\theta$ and use the definition of the Hamiltonian function $-dH = i_v\omega$, where $\omega = d\theta$:

(8.4)
$$-L_v\theta = -i_v d\theta - d i_v\theta = d(H - i_v\theta).$$

From formulas (8.3–4) one finds the derivative $d/dt\, h$:

$$\frac{d}{dt} h = H - i_v\theta.$$

(Actually, formulas (8.3–4) allow one to reconstruct the derivative up to an additive constant only, which turns out to be zero by virtue of the vanishing boundary conditions for all H, v, and h.)

Now, Theorem 8.7 follows from the following lemma.

Lemma 8.10.

$$-\int_B (i_v\theta)\,\omega^n = n\int_B H\,\omega^n.$$

Proof of Lemma. Owing to the properties of the inner derivative operator i_v, we have

$$-\int_B i_v\theta \wedge \omega^n = -\int_B \theta \wedge i_v(\omega^n)$$

$$= -n\int_B \theta \wedge i_v\omega \wedge \omega^{n-1} = n\int_B \theta \wedge dH \wedge \omega^{n-1}.$$

Moving the exterior derivative d to the 1-form θ gives

$$n\int_B d\theta \wedge H \wedge \omega^{n-1} - n\int_{\partial B} \theta \wedge H \wedge \omega^{n-1} = n\int_B H\,\omega^n,$$

since the function H vanishes on ∂B. This completes the proof of Lemma 8.10 and Theorem 8.7. $\square$

Remark 8.11. Theorem 8.7 can be reformulated as follows. Define the *Calabi integral* of a Hamiltonian function H as $\int_B H\omega^n$. This formula defines a linear function (or an exterior 1-form) on the Lie algebra $\mathrm{ham}_\partial(B)$.

The *Calabi form* on the corresponding group $\mathrm{Ham}_\partial(B)$ is the right-invariant differential form coinciding with the Calabi integral on the Lie algebra $\mathrm{ham}_\partial(B)$. The Calabi form is actually a *bi-invariant* (i.e., both left- and right-invariant) 1-form on the group of Hamiltonian diffeomorphisms $\mathrm{Ham}_\partial(B)$. It immediately follows from the fact that the Calabi integral, defined on $\mathrm{ham}_\partial(B)$, is invariant under the adjoint representation of this group $\mathrm{Ham}_\partial(B)$ in the corresponding Lie algebra $\mathrm{ham}_\partial(B)$. In turn, the latter holds because a symplectomorphism sends the Hamiltonian vector field of H to the Hamiltonian vector field of the transported function, while preserving the form ω^n. Hence, the symplectomorphism action preserves the integral.

The *Calabi invariant* $\mathrm{Cal}(g)$ of a Hamiltonian diffeomorphism g in $\mathrm{Ham}_\partial(B)$ is the integral of the Calabi form along a path g_t in $\mathrm{Ham}_\partial(B)$ joining the identity diffeomorphism with g.

Theorem 8.7′. *The Calabi form is exact: The integral depends only on the final point g and not on the connecting path.*

Although we have already proved the exactness of the Calabi form in slightly different terms, we present here a shortcut to prove its closedness. It would imply exactness if we knew that the group of Hamiltonian diffeomorphisms is simply connected, i.e., that every path connecting the identity with g in $\mathrm{Ham}_\partial(B)$ is homotopical (or at least homological) to any other. Unfortunately, we do not know whether this is the case in all dimensions (see Remark 8.9).

Proof of closedness. We start with a well-known general fact:

Lemma 8.12. *For any right-invariant differential form α on a Lie group,*

$$d\alpha(\xi, \eta) = \alpha([\xi, \eta])$$

for every pair of vectors ξ, η in the Lie algebra.

(This formula follows from the definition of the exterior differential d; see Section I.7.B). Therefore, the differential of the Calabi form is minus the integral of the Poisson bracket of two Hamiltonian functions.

Lemma 8.13. *Let H be a Hamiltonian function defined on the ball B and constant on the boundary ∂B. Then the Poisson bracket of H with another Hamiltonian function F has zero integral over B.*

Example 8.14. For two smooth functions F and H in a bounded domain D of the plane (p, q),

$$\int_D \{F, H\}\, dp \wedge dq = 0,$$

provided that H is constant on ∂D (i.e., that the Hamiltonian field of H is tangent to ∂D). Indeed,

$$\int_D \{F, H\}\, dp \wedge dq = \int_D dF \wedge dH = -\int_{\partial D} H\, dF = -H \int_{\partial D} dF = 0,$$

since H is constant on the boundary ∂D.

Proof of Lemma 8.13. The Poisson bracket $\{F, H\}$ is (minus) the derivative of F along the Hamiltonian vector field of H. Consider the $2n$-form in B that is the result of transporting the $2n$-form $F\omega^n$ by the flow of H. This flow leaves the ball B invariant, since H is constant on the boundary ∂B, and the corresponding Hamiltonian flow is tangent to ∂B.

Then the integral of the Poisson bracket $\{F, H\}$ is equal to (minus) the time derivative of the integral over B of this resulting form. But this Hamiltonian flow preserves ω^n and hence preserves the integral. Thus, the time derivative of the integral of the transported form vanishes, and so does the integral of the Poisson bracket.   □

Remark 8.15. Similarly, for any two smooth functions on a *closed* compact symplectic manifold, the integral of their Poisson bracket vanishes. Here one might replace one of the functions by a closed (nonexact) differential 1-form; it does not change the proof. Moreover, every function on a connected closed symplectic manifold whose integral vanishes can be represented as a sum of Poisson brackets of functions on this manifold [Arn7].

The closedness of the Calabi form (the invariance of integral (8.2) under the deformations of the path) follows immediately from Lemmas 8.12 and 8.13: The differential of the Calabi form is the integral of (minus) the Poisson bracket of any two functions from $\mathrm{Symp}_\partial(B)$, which always vanishes. $\square$

Now we are ready to prove the infiniteness of the diameter of the symplectomorphism group.

Proof of Theorem 8.3 for $\mathrm{Ham}_\partial(B^{2n})$**.** Let the ball B^{2n} be equipped with the standard symplectic structure, and let $\mu = \omega^n$ denote the corresponding volume form. The ℓ_2-length of a path $\{g_t\}$ in the right-invariant metric on $\mathrm{Ham}_\partial(B)$ (g_t being the flow of a time-dependent Hamiltonian function $H_t(x)$ joining the endpoints $g_0 = \mathrm{Id}$ and $g_1 = g$) is given by

$$\ell(\{g_t\}) = \int\limits_0^1 \left(\int\limits_B \|\frac{dg_t(x)}{dt}\|^2 \mu \right)^{1/2} dt$$

$$= \int\limits_0^1 \left(\int\limits_B \|\nabla H_t(g_t)\|^2 \mu \right)^{1/2} dt = \int\limits_0^1 \|\nabla H_t\|_{L_2(B)} \, dt.$$

Then the desired estimate follows from the Poincaré and Schwarz inequalities:

$$\ell(\{g_t\}) = \int\limits_0^1 \|\nabla H_t\|_{L^2(B)} \, dt \ge c_1 \int\limits_0^1 \|H_t\|_{L^2(B)} \, dt$$

$$\ge c_2 \int\limits_0^1 \|H_t\|_{L^1(B)} \, dt \ge c_2 \int\limits_0^1 \int\limits_B H_t \mu \, dt = c_2 \cdot \mathrm{Cal}(g).$$

Owing to the surjectivity of the map $\mathrm{Cal}: \mathrm{Ham}_\partial(B) \to \mathbb{R}$, one can find a Hamiltonian diffeomorphism (see Remark 8.4) with an arbitrarily large Calabi invariant and therefore arbitrarily remote from the identity. $\square$

Analogous statements for the right-invariant metric generated by the L^1-norm on vector fields $\mathrm{ham}(M)$ and for nonexact symplectomorphisms (Theorem 8.3 in full generality) require noticeably more work [ER2].

8.C. Bi-invariant metrics and pseudometrics on the group of Hamiltonian diffeomorphisms

The group $\text{Ham}(\mathbb{R}^{2n})$ of (compactly supported) Hamiltonian diffeomorphisms of the standard space $\mathbb{R}^{2n}$ (or the group of symplectomorphisms of a ball that are stationary in a neighborhood of its boundary) admits interesting *bi-invariant* metrics (see [Hof, E-P, H-Z, LaM, Plt, Don]). The right-invariant metrics discussed above are defined in terms of the norm of *vector fields*, which requires an additional ingredient, a metric on $\mathbb{R}^{2n}$. On the contrary, the bi-invariant metrics are defined solely in terms of the *Hamiltonian functions*.

Definition 8.16 [E-P]. Any L^p-norm ($1 \leq p \leq \infty$) on the space $C_0^\infty(\mathbb{R}^{2n})$ of compactly supported Hamiltonian functions assigns the *length* l_p to any smooth curve on the group $\text{Ham}(M)$. Given the Hamiltonian function $H_t \in C_0^\infty(\mathbb{R}^{2n})$ of a flow from f to g, we define

$$l_p(f, g) := \int_0^1 \|H_t\|_{L^p(\mathbb{R}^{2n})}\, dt.$$

The length functional generates a *pseudometric* ρ_p on the group $\text{Ham}(M)$ (i.e., a symmetric nonnegative function on $\text{Ham}(M) \times \text{Ham}(M)$ obeying the triangle inequality).

The pseudometrics ρ_p are *bi-invariant*. This immediately follows from invariance of the L^p-norm under the adjoint group action: The integral

$$\|H\|_{L^p(\mathbb{R}^{2n})}^p = \int_{\mathbb{R}^{2n}} |H(x)|^p\, \omega^n$$

persists under symplectic changes of the variable x. More generally, one can start with an arbitrary symplectically invariant norm on the algebra $\text{ham}(M)$.

In particular, the distance $\rho_\infty(\text{Id}, f)$ in the L^∞-(pseudo-) metric between any Hamiltonian diffeomorphism $f \in \text{Ham}(\mathbb{R}^{2n})$ and the identity element Id reads

$$\rho_\infty(\text{Id}, f) = \inf_H \int_0^1 \sup_x |H(x, t)|\, dt,$$

where the infimum is taken over all Hamiltonian functions $H(x, t)$ corresponding to flows starting at Id and ending at f. By definition, $\rho_\infty(f, g) := \rho_\infty(\text{Id}, fg^{-1})$.

This bi-invariant (pseudo-) metric ρ is equivalent to the one introduced by Hofer [Hof]:

$$(8.5) \qquad \rho_\infty'(\text{Id}, f) = \inf_H \int_0^1 (\sup_x H(x, t) - \inf_x H(x, t))\, dt.$$

We use the notation ρ_∞ in the sequel for both ρ_∞ and ρ_∞'.

Theorem 8.17 [Hof]. *The (pseudo-) metric ρ_∞ is a genuine bi-invariant metric on* Ham(M)*; i.e., in addition to positivity and the triangle inequality, the relation $\rho_\infty(f, g) = 0$ implies that $f = g$.*

Lalonde and McDuff [LaM] showed that ρ_∞, defined by the same formula (8.5) for any symplectic manifold (M, ω), is a true metric on the group Ham(M). They used it to prove Gromov's nonsqueezing theorem in full generality for maps of arbitrary symplectic manifolds into a symplectic cylinder.

It turns out, however, that the limit case $p = \infty$ is the only L^p-norm on Hamiltonians that generates a metric. For $1 \le p < \infty$ there are distinct symplectomorphisms with vanishing ρ_p-distance between them [E-P]. The features of the (pseudo-) metrics above are deduced from special properties of the following symplectic invariant, first introduced by Hofer for subsets of $\mathbb{R}^{2n}$ and called the displacement energy.

Let ρ be a bi-invariant (pseudo-) metric on the group of Hamiltonian diffeomorphisms Ham(M) of an open symplectic manifold M.

Definition 8.18. The *displacement energy $e(A)$ of a subset $A \subset M$ is the (pseudo-) distance from the identity map to the set of all symplectomorphisms that push A away from itself*: $e(A) = \inf\{\rho(\mathrm{Id}, f)\}$, where the infimum is taken over all $f \in$ Ham(M) such that $f(A) \cap A = \emptyset$. (If there is no such f, we set $e(A) := \infty$.)

Theorem 8.19 [E-P]. *Let ρ be a bi-invariant metric on* Ham(M). *Then the displacement energy of every open bounded subset $A \subset M$ is nonzero: $e_\rho(A) \ne 0$.*

For instance, for a disk $B \subset \mathbb{R}^2$ of radius R, the displacement energy in Hofer's metric is πR^2 [Hof]. Furthermore, the displacement energy is nonzero for every compact *Lagrangian submanifold* of M [Che]. (A submanifold L of a symplectic manifold (M^{2n}, ω) is called *Lagrangian* if dim $L = n$ and the restriction of the 2-form ω to L vanishes.)

Proof of Theorem 8.19. First notice that for the group commutator $[\phi, \psi]$ of any two elements $\phi, \psi \in$ Ham(M) one has

$$(8.6) \qquad \rho(\mathrm{Id}, [\phi, \psi]) \le 2 \min(\rho(\mathrm{Id}, \phi), \rho(\mathrm{Id}, \psi)).$$

This follows from the bi-invariance of the metric ρ and the triangle inequality.

Choose arbitrary diffeomorphisms $\phi, \psi \in$ Ham(M) such that their supports are in A and $[\phi, \psi] \ne$ Id. Then Theorem 8.19 will be proved with the following lemma:

Lemma 8.20. $\rho(\mathrm{Id}, [\phi, \psi]) \le 4e_\rho(A)$.

Proof of Lemma. Assume that a Hamiltonian diffeomorphism $h \in$ Ham(M) displaces A: $h(A) \cap A = \emptyset$. Then the diffeomorphism $\theta := \phi h^{-1}\phi^{-1}h$ has the

same restriction to A as ϕ. Hence, $\phi^{-1}\psi\phi = \theta^{-1}\psi\theta$. Utilizing the bi-invariance and the inequality (8.6), we have

$$\rho(\mathrm{Id}, [\phi, \psi]) = \rho(\psi, \phi^{-1}\psi\phi) = \rho(\psi, \theta^{-1}\psi\theta) \leq 2\rho(\mathrm{Id}, \theta) \leq 4\rho(\mathrm{Id}, h).$$

Minimization over h completes the proof.   □

Corollary 8.21 [E-P]. *The (pseudo-) metric ρ_p on $\mathrm{Ham}(M)$ generated by the L^p-norm on $C_0^\infty(M)$ is not a metric for $p < \infty$.*

Proof of Corollary. Let $B \subset M$ be an embedded ball and $\{g_t^H\}$ a (compactly supported) Hamiltonian flow that pushes B away from itself: $g_1^H(B) \cap B = \emptyset$. This flow is generated by a function $H \in C_0^\infty(M \times [0, 1])$. Introduce a new Hamiltonian function $K(\cdot, t)$ by smoothly cutting off $H(\cdot, t)$ outside a neighborhood $U_t \subset M$ of the moving boundary $g_t^H(\partial B)$; see Fig. 57.
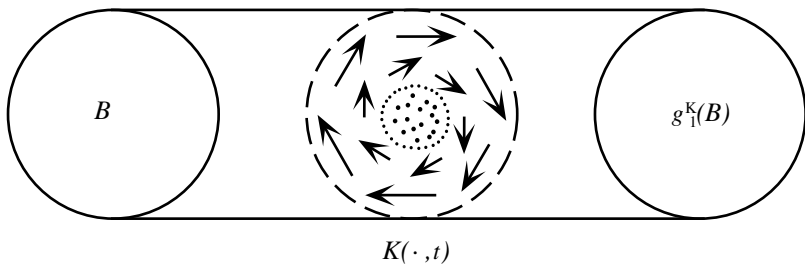
FIGURE 57.   Displacement of a ball with rotation.

The flows of K and H coincide when restricted to the boundary ∂B: $g_t^K(\partial B) = g_t^H(\partial B)$ for every t, and therefore $g_1^K(B) \cap B = \emptyset$.

Shrinking the neighborhoods U_t, one can make the L^p-norm of $K(\cdot, t)$ (and hence the distance $\rho_p(\mathrm{Id}, g_1^K)$) arbitrarily small for every $p \neq \infty$. Thus the displacement energy of B associated to ρ_p, $p \neq \infty$ vanishes, and Theorem 8.19 is applicable.   □

Informally, one can push a ball away from itself with an arbitrarily low energy, but the tradeoff is an extremely fast rotation of the shifted ball near the boundary: The function $K(\cdot, t)$ has steep slopes (and hence a large gradient) in a neighborhood of ∂B.

Let us confine ourselves to the case of compactly supported Hamiltonian diffeomorphisms in $\mathbb{R}^{2n}$.

Remark 8.22 [E-P]. For every diffeomorphism $\phi \in \mathrm{Ham}(\mathbb{R}^{2n})$ and $1 < p < \infty$, one has $\rho_p(\mathrm{Id}, \phi) = 0$. If $p = 1$, then $\rho_1(\mathrm{Id}, \phi) = |\mathrm{Cal}(\phi)|$.

The situation is completely different for the bi-invariant metrics of L^∞-type. We refer the reader to [H-Z] for an account of other peculiar properties of symplectomorphism groups.

In particular, consider the embedding of the group of Hamiltonian diffeomorphisms, say, of the ball $B \subset \mathbb{R}^{2n}$, into group of all compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$:

Theorem 8.23 [Sik]. *The subgroup of all Hamiltonian diffeomorphisms* $\mathrm{Ham}_\partial(B)$ *of a unit ball (steady near the boundary) has a finite diameter (in Hofer's metric) in the group of all compactly supported Hamiltonian diffeomorphisms of* $\mathbb{R}^{2n}$.

For the diffeomorphisms with support in the ball of radius R, the diameter is majorated by $16\pi R^2$ [H-Z]. Furthermore, for Hofer's metric the following analogue of the $S\,\mathrm{Diff}(M^3)$- and L^2- metric estimates holds (cf. Theorem 7.9):

Theorem 8.24 [Hof]. *The metric* ρ_∞ *is continuous in the* C^0*-topology: For every* $\psi \in \mathrm{Ham}(\mathbb{R}^{2n})$,

$$\rho_\infty(\mathrm{Id}, \psi) \leq 128\ (\text{diameter of }\mathrm{supp}(\psi))\ |\,\mathrm{Id} - \psi\,|_{C^0},$$

where ρ_∞ *is given by (8.5).*

The diameter result changes drastically if we consider the group $\mathrm{Ham}_\partial(B)$ by itself. For every symplectic manifold M with boundary its group of Hamiltonian diffeomorphisms $\mathrm{Ham}_\partial(M)$ stationary at the boundary has an infinite diameter in Hofer's metric (compare with Theorem 8.3 asserting the infiniteness of the diameter in the right-invariant L^2-metric). This follows from the existence of symplectomorphisms with arbitrarily large Calabi invariant, which bounds below Hofer's metric; see, e.g., [ER2]. (A more subtle statement is that the diameter of the commutant subgroup of $\mathrm{Ham}_\partial(M)$, the group of Hamiltonian diffeomorphisms with zero Calabi invariant, is also infinite; see [LaM].)

Problem 8.25. Is the diameter of the group of Hamiltonian diffeomorphisms of the two-dimensional sphere finite in Hofer's metric?

8.D. Bi-invariant indefinite metric and action functional on the group of volume-preserving diffeomorphisms of a three-fold

Though divergence-free vector fields do not have analogues of Hamiltonian functions if the dimension of the manifold is at least 3, the group of volume-preserving diffeomorphisms of a simply connected three-dimensional manifold can be equipped with a bi-invariant (yet indefinite) metric.

Let M be a simply connected compact three-dimensional manifold equipped with a volume form μ. To define a bi-invariant metric, one needs to fix on the

Lie algebra $S\,\mathrm{Vect}(M)$ of divergence-free vector fields a quadratic form that is invariant under the adjoint action of the group $S\,\mathrm{Diff}(M)$ (i.e., under a change of variables preserving the volume form). Such a form has already been introduced in Chapter III (Section III.1.D) as the *Hopf invariant* (or the helicity functional, or the asymptotic linking number) of a divergence-free vector field.

Recall that we start with a divergence-free vector field v on M^3 and define the differential two-form $\alpha = i_v\mu$, which is exact on M. The Hopf invariant $\mathcal{H}(v)$ is the indefinite quadratic form

$$\mathcal{H}(v) = \int_M d^{-1}\alpha \wedge \alpha.$$

The group $S\,\mathrm{Diff}(M)$ can be equipped with a right-invariant indefinite "finite signature" metric ρ by right translations of $\mathcal{H}$ into every tangent space on the group.

The quadratic form $\mathcal{H}(v)$ is invariant under volume-preserving changes of variables by virtue of the coordinate-free definition of $\mathcal{H}$. It follows that the corresponding indefinite metric ρ on the group $S\,\mathrm{Diff}(M)$ is bi-invariant. This metric has infinite inertia indices (∞, ∞), due to the spectrum of the d^{-1} (or curl^{-1}) operator (see [Arn9, Smo1]). The properties of this metric, apart from those discussed in Chapter III, are still obscure.

A similar phenomenon is encountered in symplectic topology (or symplectic Morse theory; see, e.g., [A-G, Arn22, Cha, Vit, Gro]). The action functional on the space of contractible loops in a symplectic manifold also has inertia indices (∞, ∞).

Remark 8.26. For a non-simply connected three-manifold M equipped with a volume form μ, the definition of the helicity invariant can be extended to null-homologous vector fields (i.e., the fields belonging to the image of the curl operator):

$$\mathcal{H}(v, w) = \int_M (i_v\mu) \wedge d^{-1}(i_w\mu).$$

Proposition 8.27. *The null-homologous vector fields form a subalgebra of the Lie algebra of divergence-free vector fields on M.*

Proof. For any two divergence-free vector fields v and w on M, their commutator $\{v, w\}$ is null-homologous: $i_{\{v,w\}}\mu = d(i_v i_w \mu)$. $\square$

Corollary 8.28. *The subgroup of volume-preserving diffeomorphisms of M corresponding to the subalgebra of null-homologous vector fields is endowed with a bi-invariant "finite signature" metric.*

The subalgebra of null-homologous vector fields is also a Lie ideal in the ambient Lie algebra of divergence-free fields. Moreover, the null-homologous vector fields

form the *commutant* (i.e., the space spanned by all finite sums of commutators of elements) *of the Lie algebra* of all divergence-free vector fields on an arbitrary compact connected manifold M^n with a volume form ([Arn7]; see also [Ban] for the symplectic case).

Remark 8.29. Consider a divergence-free vector field on a three-dimensional manifold that is exact and has a vector-potential. We can associate to this field some kind of Morse complex by the following construction. Associate to a closed curve in the manifold the integral of the vector-potential along this curve (if the curve is homologous to zero, it is the flux of the initial field through a Seifert surface bounded by our curve).

We have defined a function on the space of curves. The critical points of this function are the closed trajectories of the initial field. Indeed, if the field is not tangent to the curve somewhere, its flux through the small transverse area would be proportional to the area, and the first variation cannot vanish.

The positive and negative inertia indices of the second variation of this functional are both infinite. Indeed, in the particular case of a vertical field in a manifold fibered into circles over a surface, our functional is the oriented area of the projection curve. The latter is exactly the nonperturbed functional of the Rabinowitz–Conley–Zehnder theory; see [H-Z].

From this theory we know that the infiniteness of both indices is not an obstacle to the application of variational principles. We may, therefore, hope that the study of the Morse theory of our functional might provide some interesting invariants of the divergence-free vector field. In hydrodynamical terms these would be invariants of the class of isovorticed fields, that is, of coadjoint orbits of the volume-preserving diffeomorphism group.

The Morse index of a closed trajectory changes when the trajectory collides with another one, that is, when a Floquet multiplier is equal to 1. For the n-fold covering of the trajectory, the index changes when the Floquet multiplier traveling along the unit circle crosses an nth-root of unity. Thus, one may hope to have a rather full picture of the Morse complex at least for the curves in the total space of a circle bundle that are sufficiently close to the fibers.

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Chapter V

Kinematic Fast Dynamo Problems

Stars and planets possess magnetic fields that permanently change. Earth, for instance, mysteriously interchanges its north and south magnetic poles, so that the time pattern of the switches forms a Cantor-type set on the time scale (see [AnS]). The mechanism of generation of magnetic fields in astrophysical objects (or in electrically conducting fluids) constitutes the subject of *dynamo theory*. Kinematic dynamo theory studies what kind of fluid motion can induce exponential growth of a magnetic field for small magnetic diffusivity. Avoiding analytical and numerical results (though crucial for this field), we address below the topological side of the theory.

§1. Dynamo and particle stretching

1.A. *Fast and slow kinematic dynamos*

Definition 1.1. The *kinematic dynamo equation* is the equation

(1.1)
$$\begin{cases} \frac{\partial B}{\partial t} = -\{v, B\} + \eta \Delta B, \\ \operatorname{div} B = 0 \end{cases}$$

(for a suitable choice of units).

It assumes that the velocity field v of an incompressible fluid filling a certain domain M is known. The unknown magnetic field $B(t)$ is stretched by the fluid flow, while a low diffusion dissipates the magnetic energy. Here η is a small dimensionless parameter (representing *magnetic diffusivity*), which is reciprocal to the so-called *magnetic Reynolds number $R_m = 1/\eta$*. The bracket $\{v, B\}$ is the Poisson bracket of two vector fields (for divergence-free fields v and B in Euclidean 3-space, the latter expression can be rewritten as $-\{v, B\} = \operatorname{curl}(v \times B)$). The vector field v is supposed to be tangent to the boundary of the domain M at any time. The boundary conditions for B are different in various physical situations. For instance, the magnetic field of the Sun extends out into space, forming loops based on the Sun's surface and seen as protuberances. This magnetic field is not tangent to the boundary.

Alternatively, one can suppose that the boundary conditions are periodic (the "star" or "planet" is being replaced by the three-dimensional torus $\mathbb{R}^3/(2\pi\mathbb{Z})^3$) or, more generally, that M is an arbitrary Riemannian manifold of finite volume and Δ is the Laplace–Beltrami operator on M.

The linear dynamo equation is obtained from the full nonlinear system of magnetohydrodynamics by neglecting the feedback action of the magnetic field on the velocity field due to the Lorentz force. This is physically motivated when the magnetic field is small. The latter corresponds to the initial stage of the amplification of a "seed" magnetic field by the differential rotation.

The following question has been formulated by Ya.B. Zeldovich and A.D. Sakharov [Zel2, Sakh]:

Problem 1.2. *Does there exist a divergence-free velocity field v in a domain M such that the energy $E(t) = \|B(t)\|^2_{L^2(M)}$ of the magnetic field $B(t)$ grows exponentially in time for some initial field $B(0) = B_0$ and for arbitrarily low diffusivity?*

Consider solutions of the dynamo equation (1.1) of the form $B = e^{\lambda t}B_0(x)$. Such a field B_0 must be an eigenfunction for the (non-self-adjoint) operator $L_{v,\eta}$: $B_0 \mapsto -\{v, B_0\} + \eta\Delta B_0$ with eigenvalue $\lambda^{\mathbb{C}} = \lambda^{\mathbb{C}}(v, \eta)$. The eigenparameter $\lambda^{\mathbb{C}}$ is the *complex growth rate* of the magnetic field.

Definition 1.3. A field v is called a *kinematic dynamo* if the increment $\lambda(\eta) :=$ Re $\lambda^{\mathbb{C}}(\eta)$ of the magnetic energy of the field $B(t)$ is positive for all sufficiently large *magnetic Reynolds numbers* $R_m = 1/\eta$. The dynamo is *fast* if there exists a positive constant λ_0 such that $\lambda(\eta) > \lambda_0 > 0$ for all sufficiently large Reynolds numbers. A dynamo that is not fast is called *slow*.

There exist many possibilities for the dynamo effect in some "windows" in the range of the Reynolds numbers. In our formalized terminology, we shall *not* call such vector fields dynamos.

Remark 1.4. The existence of an exponentially growing mode of B is a property of the operator $L_{v,\eta}$, and this is why we call the velocity field v, rather than the pair (v, B), a dynamo. Kinematic dynamo theory neglects the reciprocal influence of the magnetic field B on the conducting fluid itself (i.e., the velocity field v is supposed to be unaffected by B). This assumption is justified when the magnetic field is small. The theory describes the generation of a considerable magnetic field from a very small "seed" field. Whenever the growing field gets large, one should take into account the feedback that is described by a complete system of MHD equations involving the Lorentz forces and the hydrodynamical viscosity.

The above question is reformulated now as the following

Problem 1.2′. *Does there exist a divergence-free field on a manifold M that is a fast kinematic dynamo?*

Our main interest is related to *stationary velocity fields* v in 2- and 3-dimensional domains M. There are several (mostly simplifying) modifications of the problem at hand. We shall split the consideration of the *dissipative* (realistic, $\eta \to +0$) and nondissipative (idealized, or perfect, $\eta = 0$) cases. In the idealized *nondissipative* case the magnetic field is frozen into the fluid flow, and we are concerned with the exponential growth of its energy.

In a *discrete* (in time) version of the question, one keeps track of the magnetic energy at moments $t = 1, 2, \ldots$. Instead of the transport by a flow and the continuous diffusion of the magnetic field, one has a composition of the corresponding two discrete processes at each step. Namely, given a (volume-preserving) diffeomorphism $g : M \to M$ and the Laplace–Beltrami operator $\eta\Delta$ on a Riemannian manifold M, the magnetic field B is first transported by the diffeomorphism to $B' := g_*B$, and then it dissipates as a solution of the diffusion equation $\partial B'/\partial t = \eta\Delta B'$:

$$B' \mapsto B'' := \exp(\eta\Delta)B'.$$

Problem 1.5. *Does there exist a discrete fast kinematic dynamo, i.e., does there exist a volume-preserving diffeomorphism $g : M \to M$ such that the energy of the magnetic field B grows exponentially with the number n of iterations of the map*

$$B \mapsto \exp(\eta\Delta)(g_*B),$$

as $n \to \infty$ (provided that η is close enough to 0)? The question is whether the energy of the nth iteration of B is minorated by $\exp(\lambda n)$ with a certain $\lambda > 0$ independent of η within an interval $0 < \eta < \eta_0$ for some η_0?

Other modifications of interest include chaotic flows, "periodic" versions of the dynamo problem (in which the field v on a 2- or 3-dimensional manifold is supposed to be periodic in time rather than stationary), as well as flows with various space symmetries (see [Bra, Bay1, 2, Chi2, 3, AZRS2, Sow2, Gil1, PPS, Rob]). In the sequel, we describe in detail certain sample dynamo constructions and the principal antidynamo theorems, along with their natural higher-dimensional generalizations. We shall see that the topology of the underlying manifold M enters unavoidably into our considerations.

The following remark of Childress shows that the difference between fast and slow dynamos is rather academic. Suppose that the dynamo increment $\lambda(\eta)$ decays extremely slowly, say, at the rate of $1/(\ln |\ln \eta|)$, as the diffusivity η goes to zero. (This is the case for a steady flow with saddle stagnation points, considered in [Sow1].) Though theoretically this provides the existence of only a slow dynamo, in practice, the dynamo is definitely fast: For instance, for $\eta = 1/(e^{e^3}) < 10^{-8}$ the increment $\lambda(\eta)$ is of order $1/3$, noticeably above zero.

Remark 1.6. A more general (and much less developed) dynamo setting is the so-called *fully self-consistent theory*. It seeks to determine both the magnetic field B and the (time-dependent) velocity field v from the complete system of magnetohydrodynamics equations:

$$\begin{cases} \frac{\partial B}{\partial t} = -\{v, B\} + \eta \Delta B, \\ \frac{\partial v}{\partial t} = -(v, \nabla)v + (\text{curl } B) \times B + \nu \Delta v - \nabla p, \\ \text{div } B = \text{div } v = 0, \end{cases}$$

for the fields B and v in a Euclidean domain (with standard necessary changes of symbols $\nabla, \Delta, \times$, and curl for a three-dimensional Riemannian manifold). We refer to Section I.10 and to [HMRW] for a group-theoretical treatment of magnetohydrodynamics, and to the interesting and substantial reviews [R-S, Chi2] for recent developments in both the kinematic and the fully self-consistent theories. Here we are solely concerned with the topological side of the fast kinematic dynamo mechanism.

1.B. Nondissipative dynamos on arbitrary manifolds

Unlike the dissipative ("realistic") dynamo problem, which is still unsolved in full generality, nondissipative ($\eta = 0$) dynamos are easy to construct on any manifold. First look at the case of a two-dimensional disk.

At first sight, *a nondissipative continuous-time fast dynamo on a disk* (or on a simply connected two-dimensional manifold) *is impossible*.

Pseudo-proof. Every area-preserving velocity field v on a simply connected two-dimensional manifold is Hamiltonian and can be described by the corresponding Hamiltonian function. All the orbits of the field v that are noncritical level curves of such a function are closed (Fig. 58).
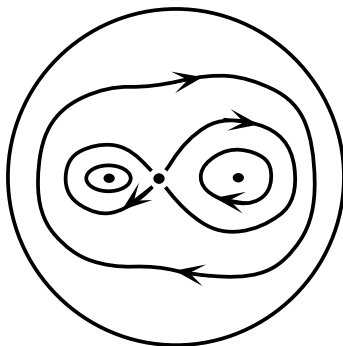
FIGURE 58. A typical Hamiltonian velocity field on a disk. Almost all orbits of the field are closed.

Consider the linearized Poincaré map along every closed orbit. The derivative g_*^T of the flow map g^T at a point of an orbit of period T is generically a Jordan

2×2 block with units on the diagonal. Indeed, the tangent vector to the orbit is mapped to itself under the Poincaré map, and hence it is eigen with eigenvalue 1. Then the Jordan block structure immediately follows from the incompressibility of the flow v, provided that it has a nondegenerate shear along the orbit (the orbit periods change with the value of the Hamiltonian).

Such a Jordan operator stretches the transported vectors of a magnetic field B linearly with the number of iterations of the Poincaré map (see Section II.5). The linear growth of the norm of B on a set of full measure implies the existence of a certain linear majorant for the increase of the energetic norm $\sqrt{E}$ over time.   □

However, one cannot neglect the contribution of the singular level sets to the magnetic energy. The following statement is folklore that directly or indirectly is assumed in any study on dynamos (see [VshM, Gil2, Koz1]).

Theorem 1.7. *On an arbitrary n-dimensional manifold any divergence-free vector field having a stagnation point with a unique positive eigenvalue (of the linearized field at the stagnation point) is a nondissipative dynamo.*

Proof. The main point of the proof is that the energy of the evolved magnetic field *inside* a small neighborhood of the stagnation point is already growing exponentially in time.

Consider the following special case: The manifold is a two-dimensional plane $M = \mathbb{R}^2$ with coordinates (x, y), while the velocity v on M is the *standard linear hyperbolic field* $v(x, y) = (-\lambda x, \lambda y)$ with $\lambda > 0$. Specify the magnetic field B to be the vertical constant field $B = (0, b)$ with support in a rectangle $R := \{|x| \le p/2, |y| \le q/2\}$; see Fig. 59.
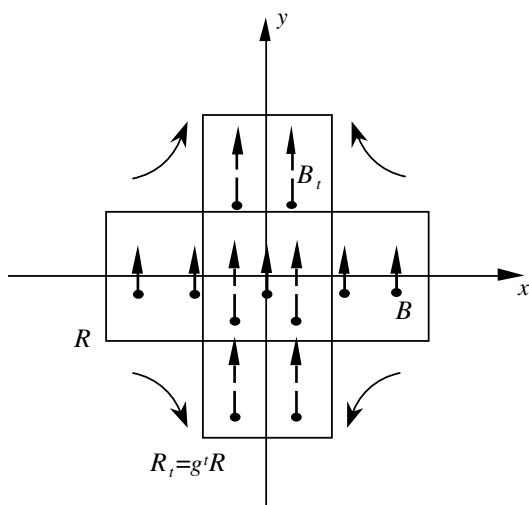
FIGURE 59. A nondissipative dynamo arising from a hyperbolic stagnation point.

At the initial moment the magnetic energy, i.e., the square of the L^2-norm of the field B, is

$$E_2(B) = \int_R B^2 \mu = pq \cdot b^2.$$

After a time period t, the image R_t of the rectangle R is squeezed in the horizontal direction by the factor $e^{\lambda t}$ and is stretched along the vertical by the same factor as is the field B as well. Then the magnetic energy of the field $B_t := g_*^t B$ is minorated by the field restriction to the initial rectangle:

$$E_2(B_t) = \int_{R_t} B_t^2 \mu > \int_{R_t \cap R} B_t^2 \mu$$

$$= (\text{area of } R_t \cap R) \cdot (e^{\lambda t} b)^2 = (pq e^{-\lambda t}) \cdot (e^{\lambda t} b)^2 = e^{\lambda t} \cdot E_2(B).$$

In turn, the latter expression $e^{\lambda t} \cdot E_2(B)$ grows exponentially with time.

The same argument applies to an arbitrary manifold M and an arbitrary velocity field v having a stagnation point with only one positive eigenvalue. One can always direct the initial magnetic field along the stretching eigenvector in some neighborhood of the stagnation point. In a cylindrical neighborhood of the stagnation point one obtains

$$E_2(B_t) = \|B_t\|_{L_2(M)}^2 \geq \|B_t\|_{L_2(R)}^2 \geq e^{\lambda t} \cdot \|B\|_{L_2(R)}^2$$

$$\geq C \cdot e^{\lambda t} \cdot \|B\|_{L_2(M)}^2 = C \cdot e^{\lambda t} \cdot E_2(B),$$

where $B_t := g_*^t B$ is the image of the field B under the phase flow of the vector field v, and C is some positive constant.   □

Remark 1.8. This gives the exponential growth of B in any L^d-norm with $d > 1$. An exponential stretching of particles (being the key idea of the above construction) will be observed in all dynamo variations below. The result is still true if the stagnation point has several positive eigenvalues, say, for a point with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0 \geq \lambda_{k+1} \geq \cdots \geq \lambda_n$, provided that $d \cdot \lambda_1 + \lambda_{k+1} + \cdots + \lambda_n > 0$, or even if the same inequality holds for the real parts of complex eigenvalues.

Even for the L^1-norm, one can provide such growth of the E_1-magnetic energy if the number of connected components of the intersection $R_t \cap R$ increases exponentially with time t. We shall observe it in the next section for the Anosov diffeomorphism of the two-torus and for any map with a Smale horseshoe.

§2. Discrete dynamos in two dimensions

2.A. Dynamo from the cat map on a torus

The main features of diversified dynamo schemes can be traced back to the following simple example (see [Arn8, AZRS1]).

Let the underlying manifold M be a two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ endowed with the standard Euclidean metric. Define a linear map $A : \mathbb{T}^2 \to \mathbb{T}^2$ to be the *cat map*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{mod } 1.$$
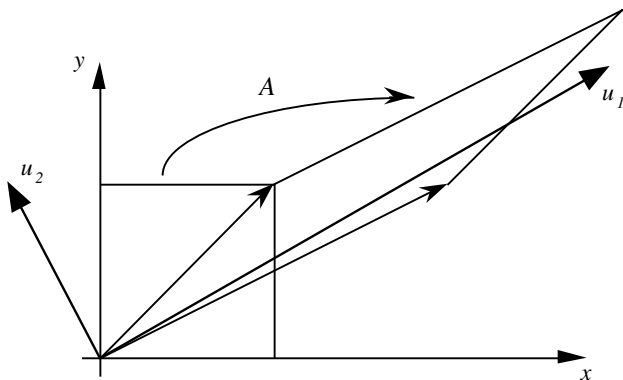
FIGURE 60.   The cat map.

The stretching (respectively, contracting) directions at all points of the torus are given by the eigenvector $u_1 \in \mathbb{R}^2$ (respectively, $u_2 \in \mathbb{R}^2$) of A, corresponding to the eigenvalue $\chi_1 = (3 + \sqrt{5})/2 > 1$ (respectively, $\chi_2 = (3 - \sqrt{5})/2 < 1$; see Fig. 60).

The constant magnetic field B_0, assuming the value $u_1 = \begin{pmatrix} 1 + \sqrt{5} \\ 2 \end{pmatrix}$ at every point of $\mathbb{T}^2$, is stretched by the factor χ_1 with every iteration of A.

A diffeomorphism $A: M \to M$ of a compact manifold M is called an *Anosov map* if M carries two invariant continuous fields of planes of complementary dimensions such that the first one is uniformly stretched and the second one is uniformly contracted. The cat map is a basic example of an Anosov map.

Remark 2.1. Taking the magnetic diffusion into account does not spoil the example of the cat dynamo. The iterations $B_{n+1} = \exp(\eta\Delta)[A(B_n)]$ with $B_0 = B$ (and $\eta \neq 0$) give the same exponential growth in spite of the diffusion. Indeed, the field B is constant, and hence the diffusion does not change the field or its iterations:

$$\|B_n\|_{L_2} = \chi_1^n \|B_0\|_{L_2}.$$

Furthermore, one can pass from a linear automorphism of the two-torus to an arbitrary smooth diffeomorphism $g : \mathbb{T}^2 \to \mathbb{T}^2$ ([Ose3]; see Section 2.C below).

Remark 2.2. The cat map $A : \mathbb{T}^2 \to \mathbb{T}^2$ provides an example of a nondissipative L^1-dynamo. It provides the exponential growth of the number of connected components in the intersection $R \cap A^n(R)$ of the rectangle R (from Theorem 1.7) with its iterations.

The cat map on the two-torus can be adjusted to produce a nondissipative dynamo action on a two-dimensional disk. The idea is the use of a ramified two-sheet covering $\mathbb{T}^2 \to S^2$, along with an Anosov automorphism of $\mathbb{T}^2$; see Fig. 61. The central symmetry of the plane $\mathbb{R}^2$ provides an involution on the torus, and its orbit space is homeomorphic to the sphere S^2. The automorphism

$$A^3 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$$

of $\mathbb{R}^2$ has four fixed points on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the points with integral and semi-integral coordinates on $\mathbb{R}^2$, and therefore it descends to the quotient space $\mathbb{T}^2/\mathbb{Z}_2 = S^2$.
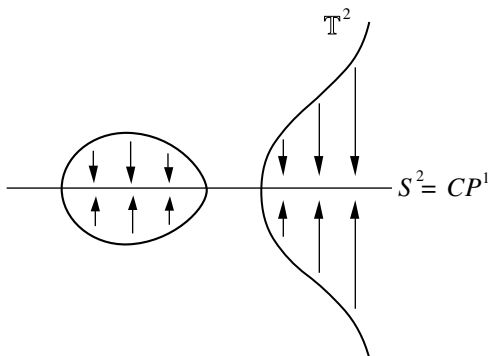
FIGURE 61.   The covering of the sphere by a torus ramified at four points. The torus $w = \sqrt{z^3 - z}$ in $\mathbb{C}P^2$ maps to the sphere $S^2 = \mathbb{C}P^1$ by the projection $(z, w) \mapsto z$.

This idea was explored as early as in 1918 by Lattes [Lat], and is rather popular now in models of ergodic theory and holomorphic dynamics [Lyub, Kat1].

In the context of dynamo theory, constructions exploiting the maps on the (non-smooth) quotient $\mathbb{T}^2/\mathbb{Z}_2$ appeared in [Gil2], along with results of numerical simulations. A substantial analysis given there shows that for the Lattes map of the disk any magnetic field after several iterations has a fine structure in which oppositely oriented vectors appear arbitrarily close to each other (Fig. 62). In the presence of diffusion the dissipation action, large at these places, inevitably prevents the rapid growth of magnetic energy.

The trick to overcoming this difficulty in three-dimensional dynamo models is to include a nontrivial shear of "different pieces" of the manifold into an iteration procedure such that diffusion averaging mostly affects the parts with the same direction of the magnetic field (see [Gil2, B-C, ChG]).
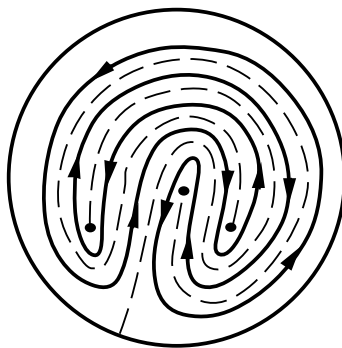
FIGURE 62.   Cancellations in the magnetic field under iterations (from [Gil2]).

There remains a possibility that a dissipative fast dynamo action in domains in $\mathbb{R}^3$ can be produced analytically, starting with the construction, known in ergodic theory, of a Bernoulli diffeomorphism on the disk.

Definition 2.3.   The *Lyapunov exponent* of a map g at a point x in the direction of a tangent vector B is the growth rate of the image length of B under the iterations of g measured by

$$\chi(x, B) := \liminf_{n \to \infty} \frac{\ln \|g_*^n B\|}{n}.$$

The Lyapunov exponents of the Lattes type diffeomorphism of the two-dimensional disk D^2 can be made positive almost everywhere (see [Kat2]). The fields of stretching directions are, in general, nonsmooth. The diffusion term of a dissipative dynamo should correspond to "random jumps of particles," in addition to the smooth evolution along the flow of v (in the spirit of [K-Y]).

2.B. Horseshoes and multiple foldings in dynamo constructions

Definition 2.4.   A phase point of a (discrete or continuous) dynamical system is said to be *homoclinic* if its trajectory has as its limits as $t \to \pm\infty$ one and the same stationary point of the system (Fig. 63).

Proposition 2.5 [Koz1]**.**   *Any area-preserving map of a surface having a homoclinic point can serve as a nondissipative two-dimensional L^1-dynamo.*

Proof.   Assume that $g : D^2 \to D^2$ is a (volume-preserving) map of a two-dimensional disk to itself having a *Smale horseshoe*. This means that there is a rectangle $R \subset D^2$ on which the map g is a composition of the following two steps. First, the rectangle is squeezed in the horizontal direction by the factor e^{λ}
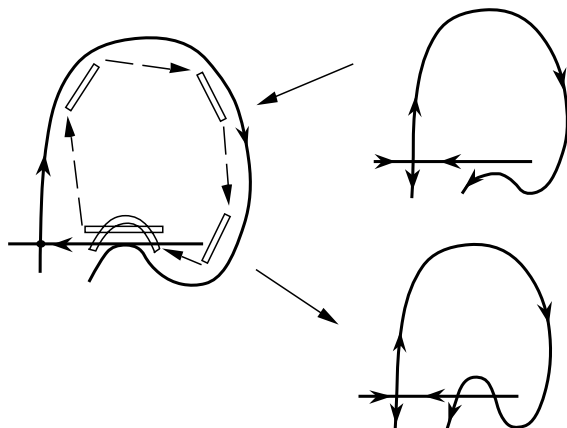
FIGURE 63.   A homoclinic point and its bifurcation.

and stretched in the vertical direction by the same factor, keeping its area the same (Fig. 64).

Then the rectangle obtained is bent in such a way that it intersects the original rectangle twice (see Fig. 64).

Under the iterations of the procedure described, the number of connected components of the intersections $(g^n R) \cap R$ grows as 2^n, where n is the number of iterations. The argument of the preceding theorem now applies to the L^1-norm of the magnetic field B. Hence, $\|B_n\|_{L^1} \geq C \cdot 2^n \|B_0\|_{L^1}$.

In a neighborhood of a homoclinic point a generic map admits a Smale horseshoe. The L^1-norm of the restriction of the field to this horseshoe grows exponentially. This completes the proof.   □

Remark 2.6. The dynamics of points in the invariant set of the horseshoe is described by means of *Bernoulli sequences* of two symbols. We put the label 0
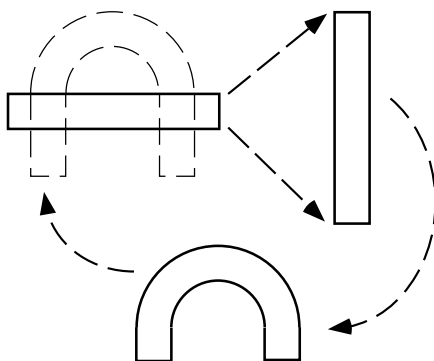
FIGURE 64.   Smale's horseshoe.

or 1 at position n if the point $g^n x$ belongs, respectively, to the left or to the right leg of the Smale horseshoe. The invariant sets of all C^2-horseshoes in a disk have measure zero [BoR]. The condition on smoothness is essential here: There is an example of a C^1-horseshoe of positive measure (see [Bow]).

We have here the same difficulty that is well known in the theory of stochastization of analytical Hamiltonian dynamical systems in a neighborhood of a periodic orbit that is the limit of the trajectory of a homoclinic point. Bifurcations of non-transversal intersections of stable and unstable manifolds of such a periodic orbit leads to the appearance of the so-called invariant set of nonwandering points. (A point a of a dynamical system g^t is called *wandering* if there exists a neighborhood $U(a)$ such that $U(a) \cap g^t U(a) = \emptyset$ for all sufficiently large t.) Though the existence of Bernoulli-type chaos on this set has been known since the classical work of Alekseev [Al], it is still unknown whether the corresponding invariant set of the phase space has positive or zero measure. The "multiple folding" occurring in such a system is basically of the same nature as the folding in nondissipative dynamo models.

We observed such a folding of the evolved magnetic field in both the horseshoe and Lattes constructions. The following theorem shows that it is unavoidable in all dynamo constructions on the disk.

Proposition 2.7 [Koz1]**.** *Let $g : D^2 \to D^2$ be a smooth volume-preserving diffeomorphism of the two-dimensional disk with the following properties. There exist an open subset $U \subset D^2$ invariant for g and a continuous oriented line field that is defined on U and invariant for g. Then the Lyapunov exponents of g vanish almost everywhere on U.*

Notice that the Lattes map allows one to construct a diffeomorphism of the disk such that the invariant set U is this disk with 3 small disks removed and the Lyapunov exponents are positive (and equal to $\ln \chi_1 = \ln(3 + \sqrt{5})/2 > 0$) on U. However, the field of the stretching directions is not oriented. This is the major obstacle to constructing a realistic dynamo on a disk: A nonzero diffusion mixes up the vectors of the magnetic field B that are oppositely oriented and hence prevents exponential growth of the field energy.

Proof of Proposition. Assume the contrary, i.e., that the Lyapunov exponents do not vanish on set U_1 that has positive measure. It is shown in [Kat2] that periodic points of g with homoclinic intersections of their stable and unstable manifolds are dense in the closure of U_1. Consider such a point x_0 and orient upwards the unstable direction at this point (Fig. 65). Then all lines defined on the unstable manifold W^u of x_0 are tangent to it and have a compatible orientation. However, if the unstable manifold W^u meets the stable manifold W^s in one direction, then it intersects W^s roughly in the opposite direction the next time, by virtue of the simple-connectedness of the disk. (On the other hand, for instance on the torus, the unstable manifold *can* intersect the stable manifold at two consecutive points

in the same direction.) Thus, the orientation of the lines oscillates and cannot be extended continuously to the point x_0.   □
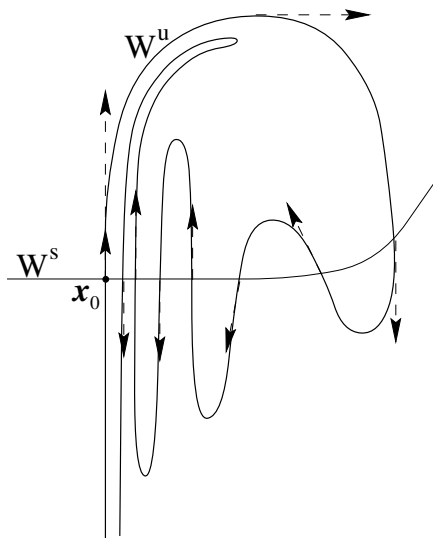
FIGURE 65.   Oriented linear elements on the unstable manifold.

In order to take into account this "mixing up" effect in the nondissipative case $(R_m = \infty)$, we introduce the following definition.

Definition 2.8. A volume-preserving diffeomorphism $g : M \to M$ of a manifold M is called a nondissipative *mean dynamo* if there exist a divergence-free vector field B and a 1-form ω such that the integral of the contraction of the form ω with the field $g_*^n B$ grows exponentially as n tends to infinity. Denote by λ_m the maximal increment of the growth:

$$\lambda_m = \sup_{\omega, B} \limsup_{n \to \infty} \frac{1}{n} \ln \left| \int_M \omega(g_*^n B) \, \mu \right|.$$

A similar definition can be introduced in the case of a vector field in place of the diffeomorphism g. The notion of a mean dynamo is stronger than that of a nondissipative L^d-dynamo $(d \geq 1)$: Any mean dynamo is a nondissipative L^d-dynamo. Another important distinction between these two concepts is the following. A sufficient condition for a nondissipative dynamo is provided by the special behavior of the diffeomorphism g in a neighborhood of a fixed point (Theorem 1.7). The situation in cases of a mean dynamo or dissipative dynamo is different. Knowing only the local behavior of g is not enough to determine whether g is a mean or a dissipative fast dynamo.

If the dimension of the manifold equals 2, a diffeomorphism g is a fast dissipative dynamo if and only if it is a mean nondissipative dynamo. In this case (dim $M = 2$)

the growth rate λ_m is determined by the operator $g_{*1} : H_1(M) \to H_1(M)$, the action of g on the first homology group of the surface M, just as in the case of the dynamo increment.

Theorem 2.9 [Koz1]. *An area-preserving diffeomorphism g of a surface M is a mean nondissipative dynamo if and only if the linear operator g_{*1} has the eigenvalue χ with $|\chi| > 1$. The mean dynamo increment λ_m is equal to $\ln|\chi|$.*

2.C. Dissipative dynamos on surfaces

Now suppose that there is a nonzero dissipation in the system. In the case of a torus, an arbitrary diffeomorphism g can be described as $g(x) = \Phi x + \psi(x)$, (x mod 1), the sum of a linear transformation $\Phi \in SL(2, \mathbb{Z})$ and a doubly periodic function ψ. In [Ose3] it is shown that for a dissipative dynamo, as $\eta \to 0$, the energy growth of a magnetic field on $\mathbb{T}^2$ is controlled solely by the matrix Φ. This matrix represents the action of g on the homology group $H_1(\mathbb{T}^2, \mathbb{R})$.

Theorem 2.10 [Ose3]. *Let $g(x) = \Phi x + \psi(x)$ be a diffeomorphism (not necessarily area-preserving) of the two-dimensional torus $\mathbb{T}^2$. Then g is a fast dissipative dynamo as $\eta \to 0$ if and only if the matrix Φ has the eigenvalue χ with $|\chi| > 1$. The dynamo increment $\lambda_0 = \lim\limits_{\eta \to 0} \lambda_\eta$ is equal to the eigenvalue $\ln|\chi|$:*

$$\lambda(\eta) = \lim_{n \to \infty} \frac{\ln\|B_n\|}{n} \to \ln|\chi| \qquad as \quad \eta \to 0,$$

for almost every initial vector field B_0.

Here

$$B_{n+1} = \exp(\eta\Delta)\left[g_* B_n\right], \quad n = 0, 1, \ldots,$$

in the area-preserving case, and

$$B_{n+1} = \exp(\eta\Delta)\left[(g_* B_n)/|\frac{\partial g}{\partial x}|\right],$$

where $|\frac{\partial g}{\partial x}|$ is the Jacobian of the map g in the non area-preserving case. The norm $\|\cdot\|$ is the L^2-norm of a vector field.

It turns out that the dynamo increment is determined exclusively by the action of g on the first homology group in the much more general situation of an arbitrary two-dimensional manifold M. For any M, each diffeomorphism $g : M \to M$ induces the linear operator g_{*i} in every vector space $H_i(M, \mathbb{R})$, the ith homology group of M, $i = 0, \ldots, \dim M$. The following statement generalizes Theorem 2.10 (and is similar to the discrete dynamos considered in Theorem 3.20).

Theorem 2.11 [Koz1]. *Let $g : M \to M$ be an area-preserving diffeomorphism of the two-dimensional compact Riemannian manifold M. Then g is a dissipative fast*

dynamo if and only if the linear operator g_{*1} has an eigenvalue χ with $|\chi| > 1$. The dynamo increment $\lambda(\eta)$ is equal to $\ln |\chi|$ and hence is independent of η:

$$\lim_{n \to \infty} \frac{\ln \|B_n\|}{n} = \ln |\chi|$$

for almost every initial vector field B_0. ($B_{n+1} = \exp(\eta\Delta)\,[g_* B_n]$, $n = 0, 1, \ldots$, and Δ is the Laplace–Beltrami operator on M.)

Remark 2.12. An eigenvalue χ with $|\chi| > 1$ exists for "most" of the diffeomorphisms of the surfaces different from the 2-sphere. Indeed, the determinant of $g_{*1} : H_1(M, \mathbb{R}) \to H_1(M, \mathbb{R})$ is equal to 1, since g is a diffeomorphism.

Proof of Theorem 2.11. First show that

$$\lim_{n \to \infty} \frac{\ln \|B_n\|}{n} \le \ln |\chi|.$$

Indeed, consider the operator $A^* = g^* \circ \exp(\eta\Delta)$ in the space of 1-forms that is L^2-conjugate to the operator $A = \exp(\eta\Delta) \circ g_*$. Let ω be its (complex) eigenvector, i.e., $A^*\omega = \kappa\omega$ with $\ln |\kappa| = \lambda(\eta)$. Such an ω exists because the norm of the conjugate operator A^* equals the norm of the operator A, and A^* is a compact operator. Note that $|\kappa| \ge 1$, since $\det |g_{*1}| = 1$. Assume that $|\kappa| > 1$ (otherwise the statement is evident).

The exterior derivative operator d commutes with g^* and with Δ. Therefore, $g^* \exp(\eta\Delta)d\omega = \kappa\,d\omega$, where g^* and Δ now act in the space of 2-forms. The pullback operator $g^* : \Omega^2(M, \mathbb{C}) \to \Omega^2(M, \mathbb{C})$ preserves the L^2-norm, while the Laplace–Beltrami operator Δ does not increase it. Hence, if $|\kappa| > 1$, it follows that the form ω is closed, $d\omega = 0$ (cf. Theorem 3.6 below).

Furthermore, the Laplace–Beltrami operator Δ does not affect the cohomology class $[\omega]$ of the closed form ω, so $g^{*1}[\omega] = \kappa[\omega]$, where g^{*1} is an action of g on the first cohomology group $H^1(M, \mathbb{C})$ containing $[\omega]$.

Therefore, either $[\omega] \ne 0$ and hence $|\kappa| \le |\chi|$ (i.e., $\lambda(\eta) \le \ln |\chi|$), or $[\omega] = 0$. In the latter case there is a function α such that $d\alpha = \omega$ and $g^* \exp(\eta\Delta)\alpha = \kappa\alpha$. The same argument as before shows that $\alpha = 0$, which contradicts the assumption that ω is an eigenvector. Thus, there remains only the possibility that $\lambda(\eta) \le \ln |\chi|$.

To show that $\lambda(\eta) \ge \ln |\kappa|$, we consider a cohomological class that is an eigenvector of g^{*1} with eigenvalue χ. Such a class is invariant under A^*, and there is an eigenvector of A^* with eigenvalue χ, so $\lambda(\eta) = \ln |\chi|$. $\square$

Theorem 2.11 holds also if g is not area-preserving, in which case

$$B_{n+1} = \exp(\eta\Delta)\left[(g_* B_n)/|\frac{\partial g}{\partial x}|\right].$$

It is easy to see that the conjugate operator has the same form as before: $A^* = g^* \exp(\eta\Delta)$.

2.D. Asymptotic Lefschetz number

The dynamo increment $\lambda(\eta)$ can also be viewed as an asymptotic version of the Lefschetz number of the diffeomorphism g (see [Ose3]).

Definitions 2.13. Let $g : M \to M$ be a generic diffeomorphism of an oriented compact connected manifold M. The *Lefschetz number $L(g)$* of the diffeomorphism g is the following sum over all fixed points $\{x_i\}$ of g:

$$L(g) = \sum_{x_i} \operatorname{sign} \det \left[\frac{\partial g}{\partial x}(x_i) - \operatorname{Id} \right],$$

where $\frac{\partial g}{\partial x}$ is the Jacobi matrix of the diffeomorphism at a fixed point and Id is the identity matrix. The *asymptotic Lefschetz number $L_{\mathrm{as}}(g)$* is

$$L_{\mathrm{as}}(g) = \limsup_{n \to \infty} \frac{1}{n} \ln |L(g^n)|$$

(in our example the lim sup is simply lim, as we shall see).

The *Lefschetz formula* relates the contribution of fixed points of the diffeomorphism g to its action on the homology groups:

$$L(g) = \sum_i (-1)^i \operatorname{Trace}(g_{*i}),$$

where the linear operators g_{*i} in the vector spaces $H_i(M, \mathbb{R})$, the ith homology group of M, are induced by the diffeomorphism $g : M \to M$.

Now the visualization of the dynamo increment $\ln |\chi|$ as the asymptotic Lefschetz number $L_{\mathrm{as}}(g)$ for $g : \mathbb{T}^2 \to \mathbb{T}^2$ (and more generally, for any $g : M \to M$) is an immediate consequence of the following rewriting of the Lefschetz formula:

$$L(g^n) = \sum_i (-1)^i \operatorname{Trace}((g^n)_{*i}) = 1 - \operatorname{Trace}(\Phi^n) + 1$$

$$= (1 - \chi^n)(1 - \chi^{-n}) = -\chi^n + O(1) \qquad \text{for} \quad |\chi| > 1, \quad n \to \infty.$$

Here we used that for $i = 0, 2$ the maps g_{*i} act identically on $H_i(M, \mathbb{R}) = \mathbb{R}$. The automorphism $g_{*1} : H_1(M, \mathbb{R}) \to H_1(M, \mathbb{R})$ can be nontrivial, and it is given by the matrix Φ in the case of a torus $M = \mathbb{T}^2$.

§3. Main antidynamo theorems

3.A. Cowling's and Zeldovich's theorems

Traditionally, necessary conditions on the mechanism of a dynamo are formulated in the form of *antidynamo theorems*. These theorems specify (usually, geometrical) conditions on the manifold M and on the velocity vector field v under which exponential growth of the L^2-norm of a magnetic vector field (or, more generally,

of any tensor field) on the manifold is impossible. In this section, the magnetic diffusivity η is assumed to be nonzero.

This direction of dynamo theory began with the following theorem of Cowling [Cow]: A steady magnetic field in $\mathbb{R}^3$ that is symmetric with respect to rotations about a given axis cannot be maintained by a steady velocity field that is also symmetric with respect to rotations about the same axis. This theorem stimulated numerous generalizations (see [Zel1, K-R, R-S]). These works show that the symmetry properties of the velocity field are irrelevant. The symmetry of the magnetic field alone prevents its growth:

Theorem 3.1. *A translationally, helically, or axially symmetric magnetic field in* $\mathbb{R}^3$ *cannot be maintained by a dissipative dynamo action.*

In what follows we shall be concerned mostly with a somewhat dual problem, in which one studies restrictions on the geometry of *velocity fields* that cannot produce exponential growth of any magnetic field.

Consider a domain in that three-dimensional Euclidean space that is invariant under translations along some axis (say, the vertical z-axis). A *two-dimensional motion* in this three-dimensional domain is a (divergence-free) horizontal vector field ($v_z = 0$) invariant under translations along the vertical axis.

Ya.B. Zeldovich considered the case where the projection of the domain to the horizontal (x, y)-plane along the vertical z-axis is bounded and simply connected.

Theorem 3.2 [Zel1]**.** *Suppose that the initial magnetic field has finite energy. Then, under the action of the transport in a two-dimensional motion and of the magnetic diffusion, such a field decays as* $t \to \infty$.

In short, "there is no fast kinematic dynamo in two dimensions."

We put this consideration into a general framework of the transport–diffusion equation for tensor densities on a (possibly non-simply connected) manifold.

3.B. Antidynamo theorems for tensor densities

Here we discuss to what extent the antidynamo theorems can be transferred to a multiconnected situation. It happens that in the non-simply connected case, instead of the decay of the magnetic field, one observes the approach of a stationary (in time) regime.

The assumption that the medium is incompressible turns out to be superfluous. In the compressible case we need merely consider the evolution of tensor densities instead of that of vector fields. The condition on the evolving velocity field v to be divergence-free can be omitted as well: We shall see that the evolution automatically leads, in the end, to a solenoidal density for an arbitrary initial condition. What really matters is the dimension of the underlying manifold.

Throughout this section we follow the paper [Arn10], to which we refer for further details.

Now we deal with an evolution of differential k-forms on a compact n-dimensional connected Riemannian manifold M without boundary. A differential k-form ω on M evolves under transport by the flow with velocity field v and under diffusion with coefficient $\eta > 0$ according to the law

$$(3.1) \qquad \frac{\partial \omega}{\partial t} + L_v \omega = \eta \Delta \omega.$$

The Lie derivative operator L_v is defined by the condition that the form is frozen into the medium. In other words, draw vectors on the particles of the medium and on their images as the particles move with the velocity field v to a new place. Then the value of the form carried over by the action (3.1) with $\eta = 0$ does not change with time when the form is evaluated on the vectors drawn.

The linear operator L_v is expressed in terms of the operator i_v (substitution of the field v into a form as the first argument) and the external derivative operator d via the *homotopy formula* $L_v = i_v \circ d + d \circ i_v$. The Laplace–Beltrami operator Δ on k-forms is defined by the formula $\Delta = d\delta + \delta d$, where $\delta = *d*$ is the operator conjugate to d by means of the Riemannian metric on M. The metric operator $* : \Omega^k \to \Omega^{n-k}$ (pointwise) identifies the k-forms on the n-dimensional Riemannian manifold with $(n - k)$-forms.

In the case of a manifold M with boundary, one usually needs specification of vanishing boundary conditions for the forms and fields.

Examples 3.3. (A) Suppose $M = \mathbb{E}^3$, Euclidean space with the metric $ds^2 = dx^2 + dy^2 + dz^2$. Specify a 2-form $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$ by choosing the vector field B with components P, Q, R; i.e., $\omega = i_B \mu$, where $\mu = dx \wedge dy \wedge dz$ is the volume element. For solenoidal fields v and B, equation (3.1) on ω results in equation (1.1) on the evolution of the magnetic field B.

(B) For functions on $M = \mathbb{R}^3$ (the case of $(k = 0)$-forms), equation (3.1) becomes the heat equation with transport:

$$(3.2) \qquad \frac{\partial f}{\partial t} = -(v, \nabla)f + \eta \Delta f.$$

(C) For a scalar density g (i.e., for $k = n$ and $\omega = g \cdot \mu$, where μ is the volume element on a Riemannian n-dimensional manifold), equation (3.1) has the form

$$(3.3) \qquad \frac{\partial g}{\partial t} = -\operatorname{div}(g \cdot v) + \eta \Delta g,$$

where the relation $d(i_\xi \mu) = (\operatorname{div} \xi) \cdot \mu$ is used.

Definition 3.4. A closed k-form ω on M is called *stationary* if it obeys the equation

$$(3.4) \qquad -L_v \omega + \eta \Delta \omega = 0.$$

Theorem 3.5 [Arn10]. *The number of linearly independent stationary k-forms is not less than the kth Betti number b_k of the manifold M.*

Recall, that the kth Betti number of M is $b_k = \dim H_k(M, \mathbb{Z})$. Examples in which the number of stationary forms is strictly larger than b_k are given below.

Theorem 3.6 [Arn10]. *If the diffusion coefficient η is large enough, then the number of linearly independent stationary k-forms is equal to the kth Betti number, and*

(a) *In each cohomology class of closed k-forms there is a stationary form.*
(b) *There is exactly one such form.*
(c) *Any closed k-form evolved according to equation (3.1) tends as $t \to \infty$ to a stationary form belonging to the same cohomology class, i.e., to a stationary form with the same integrals over every k-dimensional cycle.*
(d) *The evolution defined by equation (3.1) with any initial conditions leads in the limit to a closed form.*
(e) *All solutions of equation (3.4) are closed forms.*

Remark 3.7. Examples below show that items (b) and (c) are no longer true if the viscosity is sufficiently low (except for the cases $k = 0, 1$, or n). Exponentially growing solutions are observed for the case $k = 2, n = 3$ (which is most interesting physically; see, e.g., [AKo]) on a Riemannian manifold M, where for small diffusivity η the dimension of the space of stationary solutions is at least $2 > b_2(M) = 1$. The general Theorem 3.6 admits the following special cases:

Theorem 3.8 ($k = 0$). *For the heat equation (3.2) with transport for scalars at every positive value of the diffusion coefficient η:* (a) *every stationary solution is constant and* (b) *the solution with any initial condition tends to a constant as $t \to \infty$.*

Theorem 3.9 ($k = n$). *For the heat equation (3.3) with transport for scalar densities at every positive value of η:*

(a) *The dimension of the space of stationary solutions of equation (3.3) is equal to 1.*
(b) *There exists a unique stationary solution with any value of the integral over the entire manifold.*
(c) *The solution with any initial conditions tends as $t \to \infty$ to a stationary solution with the same integral.*
(d) *In particular, the solution with initial conditions $g = \operatorname{div} B$ converges to 0 as $t \to \infty$ regardless of the field B.*

Remark 3.10. The dynamo problem for scalar densities retains many features of the vector dynamo problem. The discussion and numerical evidence in [Bay2] show that the eigenfunctions develop a singular structure as diffusivity tends to zero.

On the other hand, the study of scalar densities (or more generally, of differential k-forms) and of their asymptotic eigenvalues allows one to prove the Morse inequalities and their generalizations by means of the method of short-wave ("quasiclassical") asymptotics [Wit1].

Theorem 3.11 ($k = 1$). *For any positive value of η in equation (3.1) for closed 1-forms:*

   (a) *The dimension of the space of stationary solutions is equal to $b_1(M)$, the one-dimensional Betti number of the manifold.*
   (b) *There exists a unique stationary solution with any given values of the integrals over independent 1-cycles.*
   (c) *The solution with any initial conditions tends as $t \to \infty$ to a stationary solution with the same integrals.*

The dynamo problem for a magnetic vector field on a compact n-dimensional Riemannian manifold is described by equation (3.3) for an $(n-1)$-form ω. The corresponding evolution of the vector density B where $\omega = i_B\mu$ is given by the law

$$\frac{\partial B}{\partial t} = -\{v, B\} - B \operatorname{div} v + \eta\Delta B.$$

Theorem 3.12 ($k = n-1$). *The divergence of the evolved density B tends to zero for every value of the diffusion coefficient $\eta > 0$. In particular, every stationary solution of equation (3.3) for $(n-1)$-forms is closed.*

Corollary 3.13. *Every solution of equation (3.1) for 1-forms on a compact two-dimensional manifold tends to a stationary closed 1-form as $t \to \infty$. For a simply connected two-dimensional manifold, every solution of equation (3.1) tends to zero (cf. Theorem 3.2).*

3.C. Digression on the Fokker–Planck equation

A problem of large-time asymptotics for scalar density transport with diffusion is already interesting in the one-dimensional case, and it arises in the study of the *Fokker–Planck equation*

$$u_t + (uv)_x = \eta u_{xx}.$$

It describes the transport of a density form $u(x)dx$ by the flow of a vector field $v(x)\partial/\partial x$ accompanied by small diffusion with diffusion coefficient η.

Suppose, for instance, that the system is periodic in x and that the velocity field v is potential. Introduce the potential U for which $v = -\,\mathrm{grad}\,U$ (the attractors of v are then the minima of U).

The stationary *Gibbs solution* of the equation has the form

$$\bar{u}(x) = \exp(-U(x)/\eta),$$

and is sketched in Fig. 66. It means that if the diffusion coefficient η is small, the density distribution is concentrated near the minima of the potential. These minima are the attractors of the velocity field v. The mass is (asymptotically) concentrated in the vicinity of the attractor, corresponding to the lowest level of the potential. (Note that the total mass is preserved by the equation: $\int u(x, t)\, dx = \text{const.}$) In the sequel, we suppose that the potential is generic and has only one global minimum.
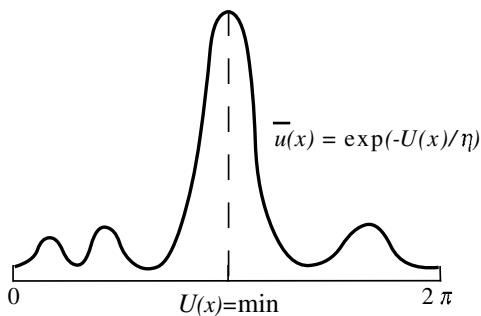
FIGURE 66.   The stationary solution of the Fokker–Planck equation.

Suppose we start with a uniformly distributed density, say, $u = 1$ everywhere. The evolution will immediately make it nonuniform, and we shall see Gauss-type maxima near all the attractors of v.

At the beginning the attractor that produces the most pronounced maximum will be the one for which the contraction coefficient (the modulus of the eigenvalue of the derivative of v at its zero point) assumes the maximal value.

Later, however, after some finite time (independent of η), the distribution will be similar to a finite set of point masses at the attractors. At this stage the most pronounced attractor will be the one collecting the largest mass. This mass, at the beginning, will be the initial mass in the basin of the attractor. Hence, in general, this attractor will be different from the one that appeared first.

The next step will consist in (slow) competition between different attractors for the masses of particles kept in their neighborhoods. This competition is (asymptotically) described by a system $\dot{m} = Am$ of linear ordinary differential equations with constant coefficients. The elements of the corresponding matrix A are the so-called *tunneling coefficients*. They are exponentially small in η, and hence the tunneling phase of the relaxation process is exponentially long ($t \sim \exp(\mathrm{const}/\eta)$). In practice, this means that in most numerical simulations one observes, instead of the limiting (Gibbs) distribution (where almost all the mass is concentrated in

one place), an intermediate distribution (concentrated in several points). This intermediate distribution evolves so slowly that one does not observe this evolution in numerical simulations.

At the end ($t \to \infty$), one of the attractors will win and attract almost all the mass. This attractor is given by the Gibbs solution, and it is somewhat unexpected: It is neither the one with the maximal initial growth of density, nor the one containing initially the most mass. In Russian, it was called the "general attractor," or the "Attractor General" (since it is as difficult to predict as it was to predict who would become the next "Secretary General").

Consider an evolution of the density (i.e., of a differential n-form) $u\mu$ on a connected compact n-dimensional Riemannian manifold with Riemannian volume element μ. The evolution under the action of a gradient velocity field $v = -\operatorname{grad} U$ and of the diffusion is described by the equation

$$u_t + \operatorname{div}(uv) = \eta \Delta u.$$

(Note that $(\operatorname{div}(uv))\mu = d(i_v(u\mu)) = L_v(u\mu)$ and $(\Delta u)\mu = (\operatorname{div} \operatorname{grad} u)\mu = d\delta(u\mu)$, since $u\mu$ is a differential n-form, and hence it is closed.)

The spectrum of the evolution operator $u \mapsto -\operatorname{div}(uv) + \eta \Delta u$ consists of a point 0 (corresponding to the Gibbs distribution), accompanied by a finite set of eigenvalues very close to 0 as $\eta \to 0$. The number of such eigenvalues is equal to the number of attracting basins of the field v, and it is defined by the Morse complex of the potential U. There is a "spectral gap" between these "topologically necessary" eigenvalues and the rest of the spectrum (which remains at a finite distance to the left of the origin as $\eta \to 0$).

The tunneling linear ordinary differential equation is the asymptotic ($\eta \to 0$) description of what is happening in the finite-dimensional space spanned by the eigenvectors corresponding to the eigenvalues close to 0. The eigenvalues are of order at most $\exp(-\operatorname{const}/\eta)$ as $\eta \to 0$, while the characteristic tunneling time is of order $\exp(\operatorname{const}/\eta)$. This explains the slow decay of the modes corresponding to the nongeneral attractors.

Remark 3.14. In spite of the evident importance of the problem, a description of the events given above does not seem to be presented in the literature (cf., e.g., [F-W]). The above description is based on an unpublished paper by V.V. Fock [Fock] and on the work of Witten [Wit1] and Helffer [Helf].[1] Fock also observed that the asymptotics of the density at a generic point of the border between the basins of two competing attractors involve a universal (erf) function in the transversal direction to the boundary hypersurface. Here, time is supposed to be large but fixed while $\eta \to 0$. The density is asymptotically given by an almost eigenfunction (quasimode) concentrated in one basin from one side of the boundary, and by the quasimode corresponding to the other basin from the other side. The transition

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[1] We thank M.A. Shubin and C. King for the adaptation of the general theory to our situation.

from one asymptotics to the other at the boundary is, according to Fock, described by the step-like "erf."

The preceding theory has an extension to the case of k-forms, where the small eigenvalues correspond to the critical points of the potential of index k (see [Wit1]).

Remark 3.15 (C. King). In the potential (one- and higher-dimensional) case, the operator $L_v - \eta\Delta$ is conjugate to a nonnegative self-adjoint operator. It shows that the spectrum is real and nonnegative.

Namely, the change of variables $\tilde{u}(x) = e^{U(x)/2\eta}u(x)$ sends the one-dimensional operator $L_v - \eta\Delta$ to the operator $\eta D_\eta^* D_\eta$, where

$$D_\eta := \frac{d}{dx} + \frac{v(x)}{2\eta} = e^{U(x)/2\eta}\frac{d}{dx}e^{-U(x)/2\eta}.$$

The latter is known as the Witten deformation of the gradient (see its spectral properties in [Helf]).

Quasiclassical asymptotics of spectra of a very general type of elliptic self-adjoint operators are treated in [Sh1] (see also [Sh2]).

Remark 3.16. The case where the velocity field is locally (but not globally) gradient is very interesting. This may already happen on the circle. In that case, the Gibbs formula $\bar{u}(x) = \exp(-U(x)/\eta)$ is meaningful only on the covering line. The potential function is no longer a periodic function, but a pseudoperiodic one (the sum of linear and periodic functions).

For every local minimum of the potential, we define the *threshold* as the minimal height one has to overcome to escape out of the well to infinity; Fig. 67. The general attractor is the one for which the threshold is maximal.
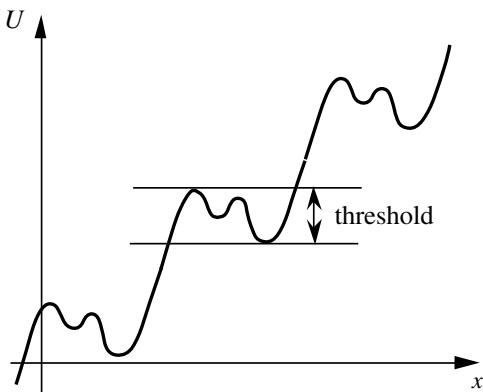
FIGURE 67.   The threshold for a local minimum of the potential is the minimal height one has to overcome to escape to infinity.

Many facts described above admit generalizations to the case of a pseudoperiodic potential in higher dimensions. In particular, the number of decaying eigenvalues is equal to the number of the field's critical points of the corresponding index [Fock].

Note that the description of the topology of pseudoperiodic functions is a rich and interesting question by itself already in two dimensions (see [Arn19, Nov3, SiK, GZ, Zor, Dyn, Pan]), where much remains to be done.

Remark 3.17. B. Fiedler and C. Rocha developed in [F-R] an interesting topological theory of the attractors of nonlinear PDEs of the type

$$u_t = f(x, u, u_x) + \eta a(x) u_{xx}.$$

They computed the Morse complex defined by the heteroclinic connections between stationary solutions in terms of some permutations and meanders. A meander is formed by a plane curve and a straight line. The corresponding permutation transforms the order of the intersection points along the straight line into their order along the curve.

3.D. *Proofs of the antidynamo theorems*

Proof of Theorem 3.5 (according to E.I. Korkina). The operator $A = -L_v + \eta\Delta$ acts on the space H of closed k-forms on M. Denote by Ker A the set of solutions of the homogeneous equation $A\omega = 0$, and by Im A the image of A in the space H. The index ind $A = \dim \text{Ker } A - \dim \text{Coker } A$, where Coker $A = H/\text{Im } A$. The index of the Laplace operator Δ is zero and so is the index of A (which differs from $\eta\Delta$ only in lower-order terms: L_v is of the first order). This means that $\dim \text{Ker } A = \dim \text{Coker } A$. But Im $A \subset \text{Im } d$ (since $A\omega = d(-i_v\omega + \eta \cdot \delta\omega)$ if $d\omega = 0$). It follows that

$$\dim(H/AH) \geq \dim(H/\{d\omega^{k-1}\}) = b_k$$

(De Rham's theorem).   □

Proof of Theorem 3.6. *(i)* The evolution defined by equation (3.1) does not affect the cohomology class of the closed form ω, since $A\omega = d(-i_v\omega + \eta \cdot \delta\omega)$ is an exact form.

(ii) For forms ω from the orthogonal complement to the subspace of harmonic forms in the space of closed forms H the following relations hold:

(3.5a) $(\omega, \omega) \leq \alpha(\delta\omega, \delta\omega),$

(3.5b) $|(\omega, L_v\omega)| \leq \beta(\delta\omega, \delta\omega),$

where α and β are positive constants independent of ω.

Indeed, (3.5a) is the Poincaré inequality (or it can be viewed as the compactness of the inverse Laplace–Beltrami operator):

$$(\omega, \omega) \le \alpha |(\Delta\omega, \omega)| = \alpha(\delta\omega, \delta\omega).$$

The inequality (3.5b) is a combination of the Schwarz and Poincaré inequalities. First note that $L_v\omega = di_v\omega$ by virtue of the homotopy formula $L_v = i_v d + di_v$ and since the form ω is closed. Then $(\omega, L_v\omega) = (\omega, di_v\omega) = -(\delta\omega, i_v\omega)$, whence applying the Schwarz inequality to the latter inner product, we get

$$|(\delta\omega, i_v\omega)|^2 \le (\delta\omega, \delta\omega)(i_v\omega, i_v\omega).$$

Now the required inequality (3.5b) follows from the above and (3.5a) in the form $(i_v\omega, i_v\omega) \le \text{const} \cdot (\delta\omega, \delta\omega)$.

(iii) From *(i)* and *(ii)* it follows that in the space of exact forms the evolution defined by equation (3.1) contracts everything to the origin if η is sufficiently large:

$$\frac{d}{dt}(\omega, \omega) = -2(\omega, L_v\omega) + 2\eta(\omega, d\delta\omega) \le 2(\beta - \eta)(\delta\omega, \delta\omega) \le -2\gamma(\omega, \omega),$$

if $\eta \ge \beta + \alpha\gamma$.

(iv) From *(i)* and *(ii)* it also follows that in an affine space of closed forms lying in one and the same cohomological class, equation (3.1) defines the flow of *contracting* transformations (in the Hilbert metric of H), and hence, it has a fixed point. This proves assertions (a)–(c).

(v) Both L_v and Δ commute with d, and therefore $d\omega$ satisfies equation (3.1) as well as ω. But the form $d\omega$ is exact, and therefore, in accordance with *(iii)*, it tends exponentially to zero as $t \to \infty$. Thus the distance between $\omega(t)$ and the space of closed forms tends exponentially to zero as $t \to \infty$.

Moreover, the same contraction to zero is observed in H^1-type metrics that take into account the derivative, provided that the diffusion coefficient η is sufficiently large (it is proved similarly to *(iii)* by using inequalities of the type

$$(\Delta\omega, L_v\Delta\omega) \le \beta(\Delta\omega, \Delta^2\omega)$$

for exact forms).

We now denote by $\omega = p + h + q$ the orthogonal decomposition of the initial form ω into exact, harmonic, and coexact (i.e., lying in the image of the operator δ) terms. Equation (3.1) assumes the form of the system

$$\dot{p} = A_1 p + A_2 h + A_3 q, \quad \dot{h} = A_4 q, \quad \dot{q} = A_5 q,$$

since for $q(0) = 0$ the form remains closed (i.e., $q(t) \equiv 0$), and a closed form retains its cohomology class (i.e., $\dot{h} = 0$ for $q = 0$).

Now, since $q(t) \to 0$ (in metrics with derivatives) exponentially, $h(t)$ tends to a finite limit (also in metrics with derivatives). But in accordance with *(iii)*, the transformation $\exp(A_1 t)$ is contracting, and hence $p(t)$ also tends to a finite limit.

Therefore, $\omega(t)$ converges to a finite limit $p(\infty) + h(\infty)$, which is a closed form. This completes the proof of assertions (d) and (e).   □

Proof of Theorem 3.8 (according to Yu.S. Ilyashenko and E.M. Landis). If the stationary solution were at any point larger than its minimum, it would immediately increase everywhere (since heat is propagated instantaneously) and would not be stationary (the so-called strengthened maximum principle). Consequently, it must be everywhere equal to its minimum; i.e., it must be constant.

The same reasoning shows that a time-periodic solution of equation (3.2) must also be a constant. Hence, the operator $A = -L_v + \eta\Delta$ on functions has no pure imaginary eigenvalues and has a single eigenvector with eigenvalue zero (by the maximum principle); this means that zero is an eigenvalue of multiplicity one and all other eigenvalues lie strictly in the left half-plane.

Since A is the sum of an elliptic operator $\eta\Delta$ and the operator $-L_v$ of lower order, we can derive by standard arguments (from the information we have obtained about the spectrum) the convergence of all solutions to constants (even in metrics with derivatives). $\qquad\square$

Proof of Theorem 3.9. The operator $A = -L_v + \eta\Delta$ on the right-hand side of equation (3.3), which sends a density g to $-\operatorname{div}(g \cdot v) + \eta\Delta g$, is conjugate to the operator $A^* = L_v + \eta\Delta$ on functions.

The eigenvalues of the operators A and A^* coincide, and therefore the dimensions of the spaces of stationary solutions of equations (3.3) and (3.2) are identical. These dimensions are equal to 1, by Theorem 3.8. Assertions (b) and (c) of Theorem 3.9 follow from the information on the spectrum of the operator A^*, that we obtained in proving Theorem 3.8. Assertion (d) follows from (c) since $\int(\operatorname{div} B)\mu = 0$. $\qquad\square$

Proof of Theorem 3.11. The operator $-L_v + \eta\Delta$ commutes with d. It follows that the solution with initial conditions $\omega_0 = df_0$ evolves under equation (3.1) in the same way as the derivative df of the solution f of equation (3.2) with initial condition f_0. From Theorem 3.8, it follows that $f \to$ const (with derivatives). This means that $df \to 0$; i.e., the exact 1-form degenerates over time. Thus the sole stationary solution that is an exact form is zero. But by Theorem 3.5, the dimension of the space of solutions of the stationary equation is not less than the first Betti number b_1, i.e., than the codimension of the subspace of exact 1-forms in the space of closed forms. Since the space of stationary solutions intersects the subspace of exact forms only at zero, its dimension is exactly equal to the Betti number b_1, and its projection onto the space of cosets of closed forms modulo exact forms is an isomorphism. This proves assertions (a) and (b). Assertion (c) follows from the fact that the exact 1-forms have vanishing integrals over all 1-cycles. $\qquad\square$

Proof of Theorem 3.12. Since d and $-L_v + \eta\Delta$ commute, the n-form $d\omega = g \cdot \mu$ evolves according to the law (3.3). By Theorem 3.9(d), the density g tends to zero as $t \to \infty$ (the condition $d\omega = g \cdot \mu$ means that g is the divergence of the vector field ξ that specifies the $(n-1)$-form $\omega = i_\xi\mu$). $\qquad\square$

Proof of Corollary 3.13. For $n = 2$, the 1-form ω is an $(n-1)$-form. By Theorem 3.12, it becomes closed ($d\omega \to 0$) as $t \to \infty$. (Here the convergence to zero is exponential even in a metric with derivatives.) Using the same reasoning as in the proof of Theorem 3.6(v), and using Theorem 3.11 to study the behavior of the exact forms, we arrive at the conclusion that the limit of ω as $t \to \infty$ exists and is closed. $\square$

3.E. Discrete versions of antidynamo theorems

Suppose that $g : M \to M$ is a diffeomorphism of a compact Riemannian manifold, $g^\star = (g^*)^{-1}$ is its action on differential forms (by forward translation), and h_η is the evolution of forms during some fixed time η under the action of the diffusion equation:

$$h_\eta := \exp(\eta\Delta), \quad f_\eta := h_\eta \circ g^\star.$$

Denote by $G^\star$ the action of $g^\star$ in the cohomology groups, $G^\star : H^*(M, \mathbb{R}) \to H^*(M, \mathbb{R})$.

Theorem 3.18 [Arn10]. (i) *The cohomology class of the closed form $f_\eta\omega$ is obtained from the class of the closed form ω by the action of $G^\star$.*
(ii) *If t is chosen sufficiently large and $G^\star$ is the identity transformation, then*

(a) *for any closed form ω the limit $\lim_{n\to\infty} f_\eta^n\omega$ exists;*
(b) *this limit is a unique closed form cohomological to ω, and it is fixed under the action of f_η;*
(c) *if the form ω is exact, then $f_\eta^n\omega \to 0$ as $n \to \infty$;*
(d) *for any form ω (not necessarily closed), the sequence of forms $f_\eta^n\omega$ is convergent as $n \to \infty$, and the limit is a closed form.*

Theorem 3.19 [Arn10]. *Let M be a two-dimensional manifold and $G^\star = \mathrm{Id}$; then assertions (a)–(d) of Theorem 3.18 are true for all $\eta > 0$ (not only for sufficiently large values of η).*

These discrete versions of Theorem 3.12 and Corollary 3.13 (and counterparts of other theorems) are proved in the same way as the original statements themselves. Moreover, one can give up the identity condition on $G^\star$. To obtain the discrete analogues of Theorems 3.6–3.12 with $G^\star \neq \mathrm{Id}$, one should not confine oneself to the stationary forms but consider the eigenvectors of the map f_η with eigenvalues $\lambda, |\lambda| > 1$. Denote by $G_k^\star$ the action of a diffeomorphism g (by forward translation) on the cohomology group $H^k(M, \mathbb{R})$ and let χ be an eigenvalue of $G_k^\star$ of maximal magnitude.

Theorem 3.20 [Koz1]. *For sufficiently large η,*

(a) *and any exact form ω, the image under iterations $f_\eta^n\omega$ tends to zero as $n \to \infty$;*

(b) *every eigenvector of f_η is a closed form;*

(c) *and a closed k-form ω, the norm $\|f_\eta^n \omega\|$ grows with the same increment as $\|(G_k^\star)^n[\omega]\|$; i.e., for all k-forms from the same cohomology class, the growth rate coincides with the growth rate of this class under the action of $G_k^\star$;*

(d) *if a cohomology class Ω is an eigenvector for the operator $G_k^\star$ with the eigenvalue χ, $G_k^\star\Omega = \chi\Omega$, then there is a form-representative $\omega \in \Omega$ such that $f_\eta\omega = \chi\omega$;*

(e) *one has for any k-form ω*

$$\lim_{n\to\infty} \frac{1}{n} \ln \|f_\eta^n \omega\| \le \ln |\chi|,$$

while for a generic k-form ω the inequality becomes equality:

$$\lim_{n\to\infty} \frac{1}{n} \ln \|f_\eta^n \omega\| = \ln |\chi|.$$

Proof. (a) can be proved using the same estimates as in the proof of Theorem 3.6(*ii*). For (b)–(e), note that the operator f_η is compact, so for any value in its spectrum there is an eigenvector. Let λ be in the spectrum and $|\lambda| > 1$. Then there is an ω such that $f_\eta\omega = \lambda\omega$. The exterior derivative d commutes with f_η, so $f_\eta d\omega = \lambda\, d\omega$, and (a) implies that $d\omega = 0$. The "diffusion" operator $\exp(\eta\Delta)$ does not change the cohomological class, so the condition $G_k^\star\Omega = \chi\Omega$ implies $f_\eta\Omega \in \Omega$. If there is a k-form $\omega \in \Omega$ such that $f_\eta\omega = \lambda\omega$, then either $\lambda = \chi$, or $[\omega] = 0$ and $|\lambda| = 1$. $\qquad\square$

The same method can be used to prove, for example, that if $M = \mathbb{T}^2$ and $G^\star = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $f_\eta^n \omega$ increases no more rapidly than the first power of n.

Remark 3.21. The case of $G^\star = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is used in [Arn8, AZRS1] to construct a fast kinematic dynamo on a three-dimensional compact Riemannian manifold; see the next section.

To the best of our knowledge, the preceding theory has not been settled for manifolds with boundary, though it certainly deserves to be.

§4. Three-dimensional dynamo models

4.A. *"Rope dynamo" mechanism*

The topological essence of contemporary dynamo constructions goes back to the following scheme proposed by Sakharov and Zeldovich (see [V-Z, ChG]) and depicted in Fig. 68.
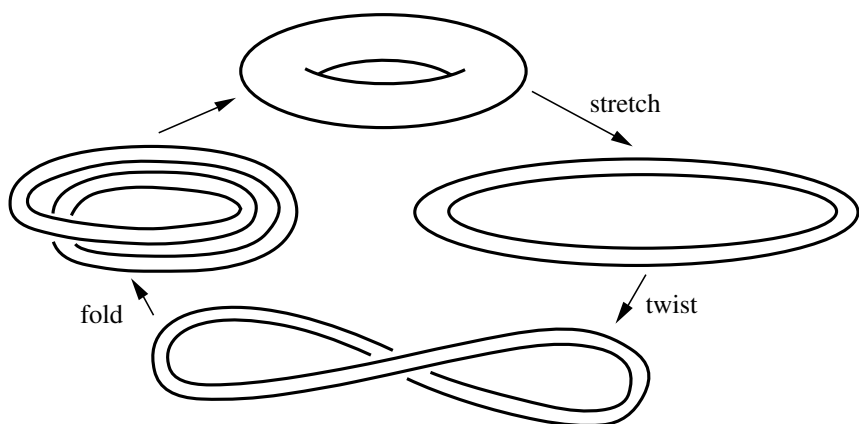
FIGURE 68.   Rope dynamo: the stretch–twist–fold mechanism.

The rapid growth of magnetic energy is achieved by iterations of the three-step transformation of a solid torus: *stretch–twist–fold*.

We start with a solid torus $S^1 \times D^2$ embedded in a three-ball. Take it out and stretch S^1 twice, while shrinking D^2 in such a way that the volume element remains preserved. Then we twist and fold the new solid torus in such a way as to obtain a twofold covering of the middle circle, and finally we put the resulting solitorus in its initial place (Fig. 68).

The energy of the longitudinal field in the solid torus (directed along the S^1 component) grows exponentially under iterations of the construction above, since the field is stretched by a factor of 2 along with the longitudinal elongation of the magnetic lines.

Though this construction is not a diffeomorphism of the solid torus onto itself, one can make it smooth, sacrificing control over stretching in a small portion of the solid torus. The loss of information about stretching of the flow in a small part of the manifold, though irrelevant for an idealized nondissipative dynamo, is essential when viscosity is taken into account.

4.B. *Numerical evidence of the dynamo effect*

The presence of chaos in *ABC* flows (see Chapter II) makes them extremely attractive for dynamo modeling. We confine ourselves to mentioning only the extensive studies in this field. The numerical and scale evidence for fast dynamo action in *ABC* and, more generally, in chaotic steady flows, can be found in, e.g., [Hen, G-F, AKo, Chi3, Bay1, Gil1, PPS] (see also [Zhel] for analogues of *ABC* flows in a three-dimensional ball).

The most extensive studies on *ABC* flows dealt with the case $A = B = C$ with the velocity field

$$\mathbf{v} = (\cos y + \sin z)\frac{\partial}{\partial x} + (\cos z + \sin x)\frac{\partial}{\partial y} + (\cos x + \sin y)\frac{\partial}{\partial z}.$$

One of the main problems in such a modeling is to estimate the increment $\lambda(\eta)$ of the fastest growing mode of the magnetic field B as a function of the magnetic diffusivity η, or of the magnetic Reynolds number $R_m = 1/\eta$. In other words, one is looking for the eigenvalue of the operator $L_{R_m} : B \mapsto -R_m\{\mathbf{v}, B\} + \Delta B$ with the largest real part. The first computations of E.I. Korkina (see [AKo]), by means of Galerkin's approximations, covered the segment of Reynolds numbers $R_m \leq 19$.

For small Reynolds numbers (i.e., for a large diffusivity η), every solution of the dynamo equation (1.1) tends to a stationary field that is determined by the cohomology class of the initial field B_0; see Theorem 3.6. Hence, for such Reynolds numbers the eigenvalue of L_{R_m} is zero independent of R_m.

When confined to the case of the fields B_0 with zero average, the largest eigenvalue of the operator L_{R_m} becomes -1 for all numbers R_m less than the critical value $R_m \approx 2.3$. The reason for this phenomenon is that $\Delta\mathbf{v} = -\mathbf{v}$ (and of course, $\{\mathbf{v}, \mathbf{v}\} = 0$), and therefore the field $\mathbf{v}$ is eigen for L_{R_m} with eigenvalue -1.

As the Reynolds number grows, there appears a pair of complex conjugate eigenvalues with $\mathrm{Re}\,\lambda = -1$. The pair of eigenvalues moves to the right and crosses the "dynamo border" $\mathrm{Re}\,\lambda = 0$ at $R_m \approx 9.0$. The increment $\mathrm{Re}\,\lambda$ stays in the right half-plane until $R_m \approx 17.5$, when it becomes negative again.

Thus the field $\mathbf{v}$ is the dynamo for $9 < R_m < 17.5$. D.J. Galloway and U. Frisch [G-F] have discovered the dynamo in this problem for $30 < R_m < 100$. It is still unknown whether this field is a fast kinematic dynamo, e.g., whether an exponentially growing mode of B survives as $R_m \to \infty$.

Numerically, the kinematic fast dynamo problem is the first eigenvalue problem for matrices of the order of many million, even for reasonable Reynolds numbers (of the order of hundreds). The physically meaningful magnetic Reynolds numbers R_m are of order of magnitude 10^8. The corresponding matrices are (and will remain) beyond the reach of any computer.

Symmetry reasoning (involving, in particular, representation theory of the group of all rotations of the cube) allows one to speed up the computations significantly. In particular, the first harmonic of some actual eigenfield for any Reynolds number can be found explicitly [Arn13]. This mode is the fastest growing for $R_m \leq 19$ as numerical experiments show.

Computer computations also suggest that the growing mode is confined to a small neighborhood of the invariant manifolds of the stagnation points, at least for $A = B = C$. There still exists a hope that this observation might lead to some rigorous asymptotic results. The asymptotic solution constructed in [DoM]

manifests concentration near the separatrices for a very long time, but not forever. No mode with such concentration was found as $t \to \infty$!

4.C. A dissipative dynamo model on a three-dimensional Riemannian manifold

In this section we consider an artificial example of a flow with exponential stretching of particles that provides the fast dynamo effect in spite of a nonzero diffusion (see [Arn8, AZRS1, 2]). In this example, everything can be computed explicitly. Its disadvantage, however, is the unrealistic uniformity of stretching and the absence of places where the directions of the growing field are opposite.

The construction is based upon the cat map of a torus, discussed above, and can be thought of as a simplified version of the model of Section II.5. In that section we considered exponential stretching of particles according to the same equations. However, unlike the magnetic field evolution in the kinematic dynamo problem, in ideal hydrodynamics the transported (vorticity) field is functionally dependent on the velocity field that is evolving it.

The domain of the flow is a three-dimensional compact manifold M that in Cartesian coordinates can be constructed as the product $\mathbb{T}^2 \times [0, 1]$ of the two-dimensional torus $\mathbb{T}^2$ with the segment $0 \leq z \leq 1$, for which the end-tori are identified by means of the transformation

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$$

(i.e., according to the law $(x, y, 0) = (2x+y, x+y, 1)$, or equivalently, $(x, y, 1) = (x - y, 2y - x, 0)$ with x mod 1, y mod 1).

To introduce a Riemannian metric on this manifold we first pass from the Cartesian coordinates x, y, z to the Cartesian coordinates p, q, z, where p has the direction of the eigenvector of A with the eigenvalue $\chi_1 = (3 + \sqrt{5})/2 > 1$, and q is directed along the eigenvector with the eigenvalue $\chi_2 = (3 - \sqrt{5})/2 < 1$. Then the metric given by the line element

$$ds^2 = e^{-2\lambda z}dp^2 + e^{2\lambda z}dq^2 + dz^2, \ \lambda = \ln \chi_1 \approx 0.75$$

is invariant with respect to the transformation A, and therefore defines an analytic Riemannian structure on the compact three-dimensional manifold M. We choose the eigenvector directions in such a way that the (p, q, z)- and (x, y, z)-orientations of $\mathbb{R}^3$ coincide.

Further, on this manifold we consider a flow with the stationary velocity field $\mathbf{v} = (0, 0, v)$ in (p, q, z)-coordinates, where $v = $ const, so that div $\mathbf{v} = 0$ and curl $\mathbf{v} = 0$. Each fluid particle moving along this field is exponentially stretched in the q-direction and exponentially contracted along the p-axis when regarded as *a particle on M*. If the magnetic Reynolds number is small (the diffusivity is large), the magnetic field growth is damped by the magnetic diffusion, and there is no dynamo effect (cf. Theorem 3.6). For small magnetic diffusivity the situation is different.

Theorem 4.1 [AZRS1]. *The vector field* **v** *defines a fast dynamo on the Riemannian manifold M for an arbitrarily small diffusivity η and in the limit η → 0. For a given initial magnetic vector field, only its Fourier harmonic independent of p and q survives and grows exponentially as t → ∞.*

Proof. Consider the following three vector fields in $\mathbb{R}^3$:

$$\mathbf{e}_p = e^{\lambda z}\partial/\partial p, \quad \mathbf{e}_q = e^{-\lambda z}\partial/\partial q, \quad \mathbf{e}_z = \partial/\partial z.$$

These fields are A-invariant, and hence they descend to three vector fields on M^3, for which we shall keep the same notations $\mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_z$. Those fields are orthogonal at every point in the sense of the above metric. Let f be a function on M; i.e., it is a function $f : \mathbb{R}^3 \to \mathbb{R}$, 1-periodic in x and y, and satisfying $f(x, y, z + 1) = f(x-y, 2y-x, z)$. Similarly, suppose that $\mathbf{B} = B_p\mathbf{e}_p + B_q\mathbf{e}_q + B_z\mathbf{e}_z$ is a (magnetic) vector field on M. Direct calculation leads to the following

Proposition 4.2. *The vector calculus formulas on M are*

$$\nabla f = \left(e^{\lambda z}\frac{\partial f}{\partial p}\right)\mathbf{e}_p + \left(e^{-\lambda z}\frac{\partial f}{\partial q}\right)\mathbf{e}_q + \left(\frac{\partial f}{\partial z}\right)\mathbf{e}_z,$$

$$\text{div}(B_p\mathbf{e}_p + B_q\mathbf{e}_q + B_z\mathbf{e}_z) = e^{\lambda z}\frac{\partial B_p}{\partial p} + e^{-\lambda z}\frac{\partial B_q}{\partial q} + \frac{\partial B_z}{\partial z}$$

(in particular, $\text{div }\mathbf{e}_p = \text{div }\mathbf{e}_q = \text{div }\mathbf{e}_z = 0$*),*

$$\text{curl}(B_p\mathbf{e}_p + B_q\mathbf{e}_q + B_z\mathbf{e}_z)$$
$$= (\text{curl}_p\,\mathbf{B})\,\mathbf{e}_p + (\text{curl}_q\,\mathbf{B})\,\mathbf{e}_q + (\text{curl}_z\,\mathbf{B})\,\mathbf{e}_z,$$

where

$$\text{curl}_p\,\mathbf{B} = e^{-\lambda z}\left(\frac{\partial B_z}{\partial q} - \frac{\partial e^{\lambda z}B_q}{\partial z}\right),$$

$$\text{curl}_q\,\mathbf{B} = e^{\lambda z}\left(\frac{\partial e^{-\lambda z}B_p}{\partial z} - \frac{\partial B_z}{\partial p}\right),$$

$$\text{curl}_z\,\mathbf{B} = e^{\lambda z}\frac{\partial B_q}{\partial p} - e^{-\lambda z}\frac{\partial B_p}{\partial q}$$

(in particular, $\text{curl }\mathbf{e}_p = -\lambda\mathbf{e}_q$, $\text{curl }\mathbf{e}_q = -\lambda\mathbf{e}_p$, $\text{curl }\mathbf{e}_z = 0$*);*

$$\Delta f = e^{2\lambda z}\frac{\partial^2 f}{\partial p^2} + e^{-2\lambda z}\frac{\partial^2 f}{\partial q^2} + \frac{\partial^2 f}{\partial z^2},$$

$$\Delta\mathbf{e}_p := -\text{curl curl }\mathbf{e}_p = -\lambda^2\mathbf{e}_p, \quad \Delta\mathbf{e}_q = -\lambda^2\mathbf{e}_q, \quad \Delta\mathbf{e}_z = 0,$$

$$\{\mathbf{e}_p, \mathbf{e}_q\} = 0, \quad \{\mathbf{e}_z, \mathbf{e}_p\} = \lambda\mathbf{e}_p, \quad \{\mathbf{e}_z, \mathbf{e}_q\} = -\lambda\mathbf{e}_q.$$

Proof of Proposition. Denote by $\phi_p = e^{-\lambda z}dp$, $\phi_q = e^{\lambda z}dq$, $\phi_z = dz$ the dual 1-forms (in $\mathbb{R}^3$ and on M). Such a form is dual to the corresponding field, in the

sense that, e.g., $\phi_p|_{\mathbf{e}_p} = 1$, $\phi_p|_{\mathbf{e}_q} = \phi_p|_{\mathbf{e}_z} = 0$, etc. Then, the expression for the differential

$$df = \frac{\partial f}{\partial p}dp + \frac{\partial f}{\partial q}dq + \frac{\partial f}{\partial z}dz = e^{\lambda z}\frac{\partial f}{\partial p}\phi_p + e^{-\lambda z}\frac{\partial f}{\partial q}\phi_q + \frac{\partial f}{\partial z}\phi_z$$

directly implies the gradient formula, etc. $\square$

The evolution (1.1) of a magnetic field $\mathbf{B} = B_p\mathbf{e}_p + B_q\mathbf{e}_q + B_z\mathbf{e}_z$ on M along the velocity field $\mathbf{v} = v\partial/\partial z$ has the following description in components:

$$\frac{\partial B_p}{\partial t} + v\frac{\partial B_p}{\partial z} = -\lambda v B_p + \eta[(\Delta - \lambda^2)B_p - 2\lambda e^{\lambda z}\frac{\partial B_z}{\partial p}],$$

$$\frac{\partial B_q}{\partial t} + v\frac{\partial B_q}{\partial z} = \lambda v B_q + \eta[(\Delta - \lambda^2)B_q + 2\lambda e^{-\lambda z}\frac{\partial B_z}{\partial q}],$$

$$\frac{\partial B_z}{\partial t} + v\frac{\partial B_z}{\partial z} = \eta(\Delta - 2\lambda\frac{\partial}{\partial z})B_z.$$

The equation for the z-component of the field splits from the rest. Suppose that the function B_z has zero average. Then, asymptotically as $t \to \infty$, the B_z-component decays (cf. Zeldovich's antidynamo theorem, Section 3.A). Indeed, the latter is the heat equation in a moving liquid. It is easy to see that B_z diminishes, since each of its maxima tends to disappear (the maximum principle). Formally, one obtains

$$\frac{d}{dt}\int B_z^2\mu = \int B_z\frac{\partial B_z}{\partial t}\mu = \eta\int B_z(\Delta B_z)\mu = -\eta\int(\nabla B_z)^2\mu.$$

Based on this, we assume in the sequel that the component B_z is constant.

It suffices to consider only one component of the vector field B, since the equations for the p- and q-components differ only by the substitution $\lambda \to -\lambda$:

(4.1) $$\frac{\partial B}{\partial t} + v\frac{\partial B}{\partial z} = \lambda v B + \eta(\Delta - \lambda^2)B,$$

where $B \equiv B_q$.

To specify the boundary conditions on B, we return to the (x, y, z)-coordinate system. Periodicity in x and y allows one to expand B into a Fourier series:

$$B(x, y, z, t) = \sum_{n,m} B_{n,m}(z, t)\exp[2\pi i(nx + my)]$$

$$= b(p, q, z, t) = \sum_{\alpha,\beta} b_{\alpha,\beta}(z, t)\exp[i(\alpha p + \beta q)],$$

where n and m are integers and α, β are related to $2\pi n$, $2\pi m$ by a linear transformation corresponding to the passage from the coordinates x, y to p, q.

Lemma 4.3. *The function $b_{0,0}(z, t)$ is periodic in z. The harmonics $b_{\alpha,\beta}(z, t)$ with $(\alpha, \beta) \neq (0, 0)$ decay exponentially in z for analytic functions B.*

Proof. Restrictions on the Fourier amplitudes come from the symmetry with respect to a shift along the z-axis: $B(x, y, z, t) = B(2x + y, x + y, z + 1, t)$. This

identity is equivalent to that on the Fourier coefficients that are acted upon by the operator conjugate to A:

$$(4.2) \qquad B_{(n,m)}(z+1,t) = B_{(n,m)A^*}(z,t).$$

Here A^* is the transpose of the matrix A, and in the case at hand $A^* = A$.

Thus the shift along the z-axis is equivalent to the transition from the Fourier amplitudes with indices (n, m) to the Fourier amplitudes with indices $(n, m)A$. Iterative applications of the matrix A shifts a typical vector (n, m) along a hyperbola in the (n, m) plane (see Fig. 69).

The only exception is the case $n = m = 0$, when the magnetic field does not depend on x, y or p, q: $(0, 0)A = (0, 0)$. Here we use that the eigendirections of A do not contain integral points (n, m) (different from $(0,0)$), since the eigenvalues of A are irrational.
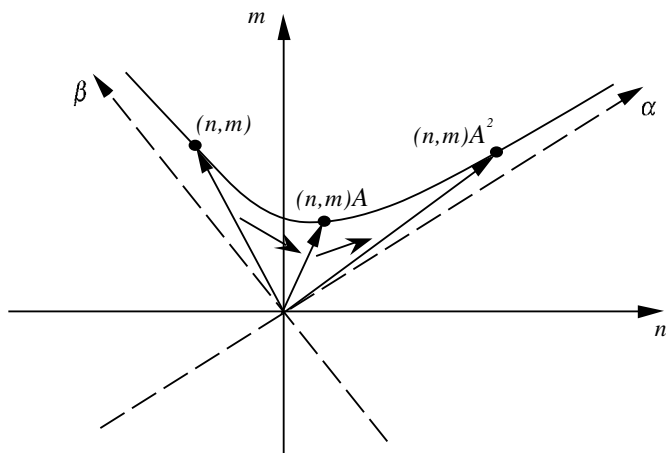
FIGURE 69.   The invariant curves for the orbits $\{(n, m)A^k\}$ are hyperbolas in the (n, m)- (or (α, β)-) plane.

On the other hand, analyticity of $B(x, y, z, t)$ implies that its Fourier harmonics $b_{\alpha,\beta}$ must decay exponentially in α and β. It follows that the functions $b_{\alpha,\beta}(z, t)$ decrease rapidly for fixed $(\alpha, \beta) \neq (0, 0)$ as $|z| \to \infty$ due to the shift property above. Periodicity in z of the zero harmonic is evident. The Lemma is proved.

$\square$

To complete the proof of Theorem 4.1 we first fix $\eta = 0$. Equation (4.1) can be solved explicitly (due to the frozenness property):

$$(4.3) \qquad b(p, q, z, t) = e^{\lambda vt} b(p, q, z - vt, 0)$$

(pass to the Lagrangian reference frame, solve the Cauchy problem, and return to the Eulerian coordinates).

Equation (4.1) may be written in the form $\partial b/\partial t = T_\eta b$, where the operator T_η (depending on the viscosity η) acts on the functions on M^3 (depending in our case on t as a parameter):

$$T_\eta b = \lambda v b + \eta(\Delta - \lambda^2) b - v \frac{\partial b}{\partial z}.$$

Consider first the nonviscous case $\eta = 0$. The nonviscous operator T_0 has a series of eigenfunctions $b_k = \exp(2\pi i k z)$, $k = 0, \pm 1, \pm 2, \ldots$, with eigenvalues $\gamma_k = \lambda v - 2\pi i k v$.

Every solution of (4.3) that does not depend on p and q (i.e., that is constant on every 2-torus $z = $ const, $t = $ const) can be represented as a linear combination of the products $b_k \cdot \exp(\gamma_k t)$ (expand (4.3) into a Fourier series in z).

The operator T_0 has no other eigenfunctions. Indeed, suppose that $b : M^3 \to \mathbb{C}$ were an eigenfunction of T_0 with an eigenvalue γ. The function $b \cdot \exp(\gamma_k t)$ would then satisfy equation (4.3). By choosing $t = 1/v$, we obtain from (4.3)

$$b(p, q, z) = e^{\lambda - \gamma} b(p, q, z - 1).$$

Using (4.2) we see that the Fourier coefficients $b_{\alpha, \beta}(z)$ along every hyperbola $\alpha = \lambda^n \alpha_0$, $\beta = \lambda^{-n} \beta_0$ form a geometric series. This contradicts the decay of the Fourier coefficients of the smooth function $b(\cdot, \cdot, z)$ on the 2-torus (unless $\alpha_0 = \beta_0 = 0$, in which case b does not depend on p and q).

The absence of eigenfunctions is explained by the continuity of the spectrum of T_0 (on the orthogonal complement to the space of functions, constant on the tori $z = $ const).

Now turn to the general case $\eta \neq 0$. As before, equation (4.1) has a sequence of solutions $b_k \cdot \exp(\gamma_k t)$, which are independent of p and q, with eigenvalues

$$\gamma_k = \lambda v + \eta(-4\pi^2 k^2 - \lambda^2) - 2\pi i k v, \quad k = 0, \pm 1, \pm 2, \ldots.$$

If η is small we find many ($\approx C\eta^{-1/2}$) growing modes. (If η is large, there is no growing mode at all, since $\operatorname{Re} \gamma_k < 0$.)

However, the behavior of the solutions whose initial field depends on p and q differs drastically from the behavior given by the frozenness condition (4.3). To explain this, consider the time evolution of b as consisting of two intermittent parts: the frozen-in stretching (4.3) ($\eta = 0$) and the pure diffusion action ($\mathbf{v} = 0$). If η is small, the stretching part might be long.

The long shift $z - vt$ ($vt \in \mathbb{Z}$) along the z-axis is equivalent to a translation (along the hyperbola) of the labels (α, β) of the harmonics $b_{\alpha, \beta}(z, t)$ for fixed z. Hence, any given harmonic will shift with time into the region of large wave numbers, where dissipation becomes important. Its amplitude will then decay in the diffusion part of the evolution. Asymptotically as $t \to \infty$, the evolving field will decay however small the viscosity η is.

Thus, we come to the conclusion that, asymptotically for $t \to \infty$, only the solution independent of p and q survives (see [AZRS1] for details on analysis of the solution asymptotics). Such a periodic in z solution in $\mathbb{R}^3$ grows exponentially in the metric ds^2 as $z \to \infty$. The increment of the corresponding exponent is

bounded away from 0 by a positive constant independent of $\eta > 0$. Finally, due to the linear relation between shifts in the z and t directions, one obtains the same exponential growth of the solution as $t \to \infty$.   □

4.D. Geodesic flows and differential operations on surfaces of constant negative curvature

Every compact Riemann surface can be equipped with a metric of constant curvature. This curvature is positive for a sphere, vanishes for a torus, and is negative for any surface with at least two handles (i.e., for any surface of genus ≥ 2).

In this section we show that the geodesic flow on every Riemann surface whose curvature is constant and negative provides an example of a fast (dissipative) kinematic dynamo. More precisely, let M^3 be the bundle of unit vectors over such a surface $P \colon M^3 = \{\xi \in TP \mid \|\xi\| = 1\}$. The geodesic flow defines a dynamical system on this three-dimensional manifold M^3 with exponential stretching of particles of M, similar to the example above. Avoiding repetition, we present here the basic formulas for the key differential operations on the bundle of unit vectors over P.

First of all, let us pass from the surface P to its universal covering $\tilde{P}$. Every such surface of constant negative curvature is covered by the Lobachevsky plane $\tilde{P} = \Lambda$, where the covering is locally isometric (that is, respecting the metrics on both spaces).

Remark 4.4. Sometimes it is convenient to think of the bundle of unit vectors $V^3 := T_1\Lambda$ over the Lobachevsky plane Λ as the group $SL(2, \mathbb{R})$. Then the space M^3 is the quotient of $SL(2, \mathbb{R})$ (or, more generally, of the universal covering $\widetilde{SL(2, \mathbb{R})}$) over a discrete uniform subgroup Γ:

$$M^3 = SL(2, \mathbb{R})/\Gamma.$$

We will deal with the following three "basic" flows on the Lobachevsky plane: the geodesic flow and two horocyclic flows. Introduce the natural coordinates (x, y, φ) in the space of line elements (or of unit vectors) $V^3 = T_1\Lambda^2$, where the Lobachevsky plane is the upper half-plane $\Lambda^2 = \{(x, y) \mid y > 0\}$ equipped with the metric $ds^2 = (dx^2 + dy^2)/y^2$, and $\varphi \in [0, 2\pi)$ is the angle of a line element with the vertical in Λ^2 (see Fig. 70a). The x-axis is called the *absolute* of the Lobachevsky plane. Recall that the geodesics in Λ^2 are all semicircles and straight lines orthogonal to the absolute (Proposition IV.1.3).

Definition 4.5. The *geodesic flow* of the Lobachevsky plane is the flow in the space of unit line elements $V^3 = T_1\Lambda^2$ that sends, for time t, every element l into the line element on Λ^2 tangent to the same geodesics as l, but at the distance t (in the Lobachevsky metric) ahead of l.
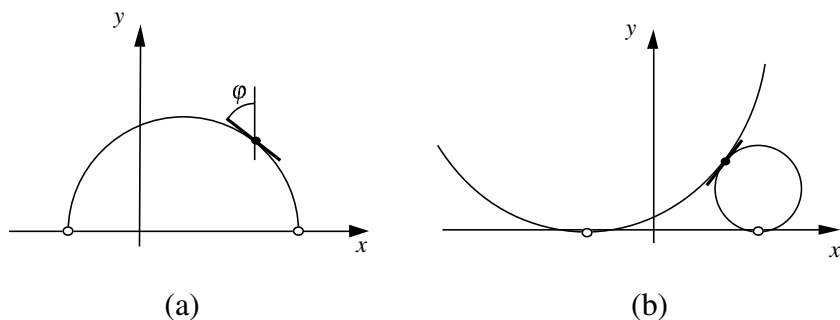
FIGURE 70. (a) Coordinates (x, y, φ) in the space of line elements along the geodesics in the Lobachevsky plane. (b) Two horocycles passing through one line element.

The limit of a sequence of Euclidean circles tangent to each other at a given point and of increasing radius in the Lobachevsky plane is called a *horocycle*.

Proposition 4.6 (see, e.g., [Arn15]). *The horocycles in the Lobachevsky plane are exactly the Euclidean circles tangent to the absolute and the straight lines parallel to it.*

Every line element (point with a specified direction) on Λ^2 belongs to two horocycles, "upper" and "lower"; see Fig. 70b.

Definition 4.5′. The *first (+)* and *second (−) horocyclic flows* on Λ^2 are the flows sending in a time t every line element on the Lobachevsky plane to the line element belonging to the same lower and upper horocycles respectively, and lying distance t ahead of it.

Explicitly, the flows are given by the following vector fields e (for the geodesic flow), h^- (for the "lower" horocyclic flow), and h^+ (for the "upper" horocyclic one) on V:

$$e = -y \sin \varphi \frac{\partial}{\partial x} + y \cos \varphi \frac{\partial}{\partial y} + \sin \varphi \frac{\partial}{\partial \varphi},$$

$$h^- = -y \cos \varphi \frac{\partial}{\partial x} - y \sin \varphi \frac{\partial}{\partial y} + (\cos \varphi - 1) \frac{\partial}{\partial \varphi},$$

$$h^+ = -y \cos \varphi \frac{\partial}{\partial x} - y \sin \varphi \frac{\partial}{\partial y} + (\cos \varphi + 1) \frac{\partial}{\partial \varphi}.$$

Proposition 4.7. *The vector fields e, h^+, h^- generate the Lie algebra $\gamma sl(2, \mathbb{R})$.*

Proof. $\{e, h^+\} = h^+$, $\{e, h^-\} = -h^-$, $\{h^+, h^-\} = 2e$, where $\{ \ , \ \}$ means the Poisson bracket of two vector fields: $L_{\{u,v\}} = L_u L_v - L_v L_u$. In coordinates it is $\{u, v\} = (u, \nabla)v - (v, \nabla)u$; see Section I.2.   □

Notice that the difference of the horocyclic fields $f := \frac{1}{2}(h^- - h^+)$ is the rotation field $f = \frac{\partial}{\partial\varphi}$. Introduce also the sum field $\tilde{e} := \frac{1}{2}(h^+ + h^-)$. The Poisson brackets between the fields e, $\tilde{e}$, f are $\{f, e\} = \tilde{e}$, $\{\tilde{e}, e\} = f$, $\{\tilde{e}, f\} = e$.

Now we define in $V^3 = T_1\Lambda^2$ a one-parameter family of metrics:

$$(4.4) \qquad d\ell^2 = \frac{dx^2 + dy^2}{y^2} + \lambda^2\left(d\varphi + \frac{dx}{y}\right)^2.$$

Proposition 4.8. *The above metrics on the space of line elements $V^3 = T_1\Lambda^2$ are singled out by the following three conditions:*

(1) *Consider the planes defining the standard Riemannian connection in $T_1\Lambda^2$ related to the Lobachevsky metric on Λ^2. The condition on a metric in $T_1\Lambda^2$ is that the fibers of the projection $T_1\Lambda^2 \to \Lambda^2$ are orthogonal to those planes.*

(2) *The above projection sends the planes to the tangent spaces to Λ^2 isometrically.*

(3) *The metrics are invariant with respect to isometries of the Lobachevsky plane.*

Proof. From (2) one can see that

$$d\ell^2 = \frac{dx^2 + dy^2}{y^2} + \lambda^2(d\varphi + a\,dx + b\,dy)^2.$$

Utilize condition (1) in the following form: The coefficients a and b obey the relation $d\varphi + a\,dx + b\,dy = 0$ along two curves in $V = T_1\Lambda^2$. One of the curves is the parallel transport of a line element along its geodesics (i.e., it is the orbit of e), and the other curve is obtained by the parallel transport of the same line element in the perpendicular direction (i.e., along the orbit of $\tilde{e}$). The calculation can be carried out at $x = 0$, $y = 1$, $\varphi = 0$, and extended by invariance due to (3). □

For each metric of the family (4.4), the basis e, $\tilde{e}$, f is orthogonal: $(e, \tilde{e}) = (e, f) = (\tilde{e}, f) = 0$; and moreover, $(e, e) = (\tilde{e}, \tilde{e}) = 1$, $(f, f) = \lambda^2$. In this normalization the volume element spanned by the three fields is $\tau(e, \tilde{e}, f) = \lambda$.

Proposition 4.9. *All the fields e, $\tilde{e}$, f, h^+, h^- are divergence-free and null-homologous. Their vorticities are as follows:*

$$\operatorname{curl} e = -\frac{e}{\lambda}, \quad \operatorname{curl} \tilde{e} = -\frac{\tilde{e}}{\lambda}, \quad \operatorname{curl} f = \lambda f, \quad \operatorname{curl} h^\pm = -\frac{\tilde{e}}{\lambda} \mp \lambda f.$$

The helicities of both of the horocyclic flows are zero, while the helicity of the geodesic flow e in the compact manifold $M^3 = T_1 P^2$ is

$$\mathcal{H}(e) = -8\pi^2\lambda^2\left((\text{genus of } P^2) - 1\right).$$

Proof. Introduce 1-forms α, $\tilde{\alpha}$, β dual to the fields e, $\tilde{e}$, f, respectively (i.e., $\alpha|_e = 1$, $\alpha|_{\tilde{e}} = \alpha|_f = 0$, etc). Now, the calculation of the vorticity and divergence

is a straightforward application of the formula for the differential of a 1-form:

$$d\gamma(v_1, v_2) = \gamma(\{v_2, v_1\}) + L_{v_1}\gamma(v_2) - L_{v_2}\gamma(v_1).$$

For instance, combining it with the formulas for Poisson brackets

$$\{e, f\} = -\tilde{e}, \quad \{e, \tilde{e}\} = -f, \quad \{\tilde{e}, f\} = e,$$

one obtains $d\alpha(e, \tilde{e}) = d\alpha(e, f) = 0$, $d\alpha(\tilde{e}, f) = -1$. Therefore, $-\lambda d\alpha = i_e\tau$. By definition, this means that div $e = 0$ and curl $e = -e/\lambda$ (see Section III.1 for more detail).

The helicity expression for the geodesic field e on $M^3 = T_1P^2$ is

$$\mathcal{H}(e) = \int_M (e, \mathrm{curl}^{-1} e)\tau = -\lambda \int_M \tau = -2\pi\lambda^2 \cdot (\text{area of } P^2).$$

Here the volume of the bundle M is the product of the fiber length $2\pi\lambda$ and the area of the surface P^2. The Gauss–Bonnet theorem reduces the area of the surface P^2 (with constant curvature) to the number of handles:

$$\text{area of } P^2 = 4\pi(\text{genus of } P^2 - 1).$$

We leave to the reader the helicity calculations for horocyclic flows $h^{\pm}$ on M.

$\square$

Returning to hydrodynamics, we immediately obtain the following

Corollary 4.10. *The velocity fields e, $\tilde{e}$, f, as well as all linear combinations of e and $\tilde{e}$, are stationary solutions of the Euler equation on $M^3 = T_1P^2$. They are also the stationary solutions of the corresponding Navier–Stokes equation on M for the vorticity field $\omega = \mathrm{curl}\, v$:*

$$\frac{\partial\omega}{\partial t} + \{\mathrm{curl}^{-1}\omega, \omega\} = -\eta \cdot \mathrm{curl}\,\mathrm{curl}\,\omega + R,$$

where R (the curl of the external force) is proportional to ω.

Proof. $\{e, \mathrm{curl}^{-1} e\} = \{\tilde{e}, \mathrm{curl}^{-1}\tilde{e}\} = \{f, \mathrm{curl}^{-1} f\} = 0.$ $\square$

The symmetry of this corollary and of the formulas of Proposition 4.9 under the interchange of e and $\tilde{e}$ is not surprising, since the flow of the field $f = \partial/\partial\varphi$ is an isometry of $V^3 = T_1\Lambda^2$, and it takes the field e to the field $\tilde{e}$ for the time $\pi/2$.

Proposition 4.11. *Every steady solution $\omega_0 = Ae + B\tilde{e}$ of the Euler equation for vorticity is unstable in the linear approximation.*

Proof. The linearized Navier–Stokes equation (cf. the linearized Euler equation (II.5.1)) for variations of velocity $v = v_0 + v_1$ and vorticity $\omega = \omega_0 + \omega_1$ is

$$\frac{\partial\omega_1}{\partial t} + \{v_0, \omega_1\} + \{v_1, \omega_0\} = -\eta\,\mathrm{curl}\,\mathrm{curl}\,\omega_1.$$

For the initial vorticity $\omega_0 = Ae + B\tilde{e}$ we have $v_0 = -\lambda\omega_0$, and hence

$$(4.5) \qquad \frac{\partial \omega_1}{\partial t} = \{\omega_0, v_1 + \lambda\omega_1\} - \eta \operatorname{curl} \operatorname{curl} \omega_1,$$

where $v_1 = \operatorname{curl}^{-1} \omega_1$. Consider the three-dimensional space of special ("long-wave") perturbations

$$\omega_1 = a\,e + b\,\tilde{e} + c\,f, \qquad v_1 = -\lambda a\,e - \lambda b\,\tilde{e} + \frac{c}{\lambda}\,f.$$

The operator on the right-hand side of formula (4.5) maps this space (with the basis e, $\tilde{e}$, f) into itself, and it is represented by the matrix

$$\begin{pmatrix} -\eta/\lambda^2 & 0 & B\xi \\ 0 & -\eta/\lambda^2 & -A\xi \\ 0 & 0 & -\eta\lambda^2 \end{pmatrix}, \qquad \text{where} \quad \xi = \lambda + \frac{1}{\lambda}.$$

Therefore, for nonzero viscosity η the eigenvalues are negative, and the corresponding modes decay. However, for $\eta = 0$ one has linear growth of the perturbations (in the direction perpendicular to ω_0 in the plane $(e, \tilde{e})$).

Remark 4.12. It is natural to conjecture that our linearized equation for $\eta = 0$ has exponentially growing solutions, and even an infinite-dimensional space of those (it has not been proved). Indeed, at least for fast-oscillating solutions, one may neglect the second term $\{v_1, \omega_0\}$ and take into account only the first one $\{v_0, \omega_1\}$, since for such solutions $v_1 = \operatorname{curl}^{-1} \omega_1$ is small compared to ω_1. Then one obtains the equation of a frozen transported field, and it has exponentially growing solutions (directions h^- or h^+ for $\omega_0 = -e$ or $\omega_0 = e$, respectively).

One can also argue that for small positive η equation (4.5) has many exponentially growing solutions.

In a similar way one can study the stationary solution $\omega_0 = f$, $v_0 = f/\lambda$. In this case the first term on the right-hand side of equation (4.5) has the form $\{\omega_0, v_1 - \omega_1/\lambda\}$. The matrix of the evolution operator for the "long-wave" perturbations is

$$\begin{pmatrix} -\eta/\lambda^2 & \xi & 0 \\ -\xi & -\eta/\lambda^2 & 0 \\ 0 & 0 & -\eta\lambda^2 \end{pmatrix}.$$

Therefore, the eigenvalues in this case are always negative for $\eta > 0$, while for $\eta = 0$ the eigenvalues are purely imaginary, $\pm i\xi$, and 0. The "fluid motion" on M corresponding to the field f is "rigid" (i.e., an isometry) and apparently stable.

As usual, the problem simplifies as we pass to the dynamo equations, where the magnetic field is not related to the velocity. Consider, for instance, the velocity field $v = Ae + B\tilde{e} + Cf$, where A, B, C are constants. The three-dimensional space of "long-wave" magnetic fields $\mathbf{B} = ae + b\tilde{e} + cf$ is invariant with respect to stretching by the flow of v, as well as with respect to "diffusion." The evolution

of a "long-wave" field is given by the matrix

$$\begin{pmatrix} -\eta/\lambda^2 & C & -B \\ -C & -\eta/\lambda^2 & A \\ -B & A & -\eta\lambda^2 \end{pmatrix}.$$

Let us confine ourselves to the case $B = C = 0$, where the particles are stretched by the geodesic flow. In this case one readily evaluates the eigenvalues of the matrix and obtains the following

Corollary 4.13. *For sufficiently small magnetic diffusion ($\eta < |A|$), the geodesic flow is a fast dynamo. The growing mode is a linear combination of the horocyclic flows (or of the flows $\tilde{e}$ and f). The growth rate (i.e., the increment of the growing mode) depends continuously on the magnetic viscosity η and tends to $|A|$ as $\eta \to 0$.*

The velocity field $\tilde{e}$ (corresponding to $A = C = 0$) shares the analogous dynamo properties.

Hypothetically, the number of exponentially growing modes in these cases increases without bound as the magnetic viscosity η tends to 0. On the other hand, for the "rigid" field f (i.e., for $A = B = 0$) one has the matrix

$$\begin{pmatrix} -\eta/\lambda^2 & C \\ -C & -\eta/\lambda^2 \end{pmatrix},$$

which indicates the absence of growth of the "long-wave" fields. Furthermore, for nonzero magnetic viscosity η the fields decay, since the matrix eigenvalues have negative real parts.

4.E. Energy balance and singularities of the Euler equation

Proposition 4.14. *If a vector field ω_0 is a solution of the following equation,*

(4.6) $\{\text{curl}^{-1}\,\omega_0, \omega_0\} = \text{const} \cdot \omega_0,$

then the constant is zero.

Proof. We look for solutions of the Helmholtz equation $\dot{\omega} = -\{v, \omega\}$, $v = \text{curl}^{-1}\,\omega$ in the form $\omega(t) = a(t)\omega_0$, where ω_0 satisfies the relation above, and $a(t)$ depends on t only. Then substitution gives the following equation in a: $\dot{a} = -a^2 \cdot \text{const}$. All nontrivial solutions of the latter equation go to infinity at finite time if the constant is nonzero. The unbounded growth of ω contradicts the energy conservation law $\dot{E} = 0$ for kinetic energy $E = \frac{1}{2} \int v^2 \mu$.   □

Among the "long-wave" vector fields on $M = T_1 P^2$ studied above, only the fields $\omega_0 = ae + b\tilde{e}$ (which commute with $\text{curl}^{-1}\,\omega_0$) satisfy equation (4.6). Indeed, for $\omega_0 = ae + b\tilde{e} + cf$ one gets from the commutation relations discussed

$$\{\text{curl}^{-1}\,\omega_0, \omega_0\} = ac\xi\tilde{e} - bc\xi e,$$

where $\xi = \lambda + 1/\lambda$. There is no f on the right-hand side, which implies that $c = 0$.

§5. Dynamo exponents in terms of topological entropy

5.A. *Topological entropy of dynamical systems*

We have seen in Section 1.B that the exponential growth of the L^2-magnetic energy (and more generally, of the L^q-energy for $q > 1$) can be easily achieved in a nondissipative dynamo model whose velocity field has a hyperbolic stagnation point or a hyperbolic limit cycle. However, the class of nondissipative dynamos providing the exponential growth of the L^1-energy of a magnetic field is much more subtle. To specify this class, as well as to formulate the conditions for realistic (dissipative) dynamos, we need the notion of entropy of a flow or of a diffeomorphism.

Definition 5.1. Let dist be the metric on a compact metric space M, and let $g : M \to M$ be a continuous map. For each $n = 0, 1, 2, \ldots$, define a new metric $\text{dist}_{g,n}$ on X by

$$\text{dist}_{g,n}(x, y) = \max_{i=0,1,\ldots,n} \text{dist}(g^i x, g^i y).$$

A set is said to be (n, ϵ)-*spanning* if in the $\text{dist}_{g,n}$-metric, the ϵ-balls centered at the points of the set cover the space M. Let $N(n, \epsilon, g)$ be the cardinality of the minimal (n, ϵ)-spanning set. Then the *topological entropy of the map g* is defined by

$$h_{\text{top}}(g) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln N(n, \epsilon, g).$$

The *topological entropy $h_{\text{top}}(v)$ of a vector field v* is the topological entropy of the time 1 map of its flow.

One can give such a definition for an arbitrary compact topological space by replacing ϵ-balls with an open covering and maximizing over all coverings; see, e.g., [K-Y].

To visualize this notion, think of the trajectories of two points x and y as being indistinguishable if the images $g^i(x)$ and $g^i(y)$ are ϵ-close for each $i = 0, \ldots, n$. Then $N(n, \epsilon, g)$ measures the number of trajectories of length n for the diffeomorphism g that are pairwise distinguishable for given ϵ. Intuitively, positivity of entropy indicates that this number grows exponentially with n.

5.B. Bounds for the exponents in nondissipative dynamo models

Theorem 5.2 [Koz2, K-Y]. *Let v be a divergence-free C^∞-vector field on a compact Riemannian three-dimensional manifold M, and μ the Riemannian volume form on M. Assume that a magnetic field B_0 is transported by the flow g_v^t of the field v: $B(t) := g_{v*}^t B_0$. Then for every continuous field B_0 on M the increment of the L^1-growth rate is majorized by the topological entropy $h_{\text{top}}(v)$ of the field v:*

$$\limsup_{t\to\infty} \frac{1}{t} \ln \int_M \|B(t)\|\mu \le h_{\text{top}}(v).$$

Moreover, the increment is exactly equal to the topological entropy for any generic magnetic field B_0 (i.e., for a field from an open and dense subset in the space of vector fields).

The formulation of Theorem 5.2 allows one to regard it as a naive definition of topological entropy: Choose a vector field, act on it by the flow, and estimate the corresponding rate of change of the L^1-energy. To be sure that the chosen field B_0 is "generic," one can start with a pair of vector fields B_1 and B_2 that along with the field v form a basis in (almost) every tangent space of M. Then $h_{\text{top}}(v)$ is equal to the biggest rate of L^1-energy growth of these two vector fields under iterations:

$$h_{\text{top}}(v) = \max\{\lambda_1, \lambda_2\}, \qquad \text{where} \quad \lambda_i = \limsup_{t\to\infty} \frac{1}{t} \ln \int_M \|B_i(t)\|\mu.$$

Furthermore, the topological entropy of a smooth map $g : M \to M$ is equal to the maximum of the L^1-growth rate under iterations of g of a generic differential form. More precisely,

$$h_{\text{top}}(v) = \lim_{n\to\infty} \frac{1}{n} \ln \int_M \|Dg^{n*}\|\mu,$$

where Dg^{n*} is a mapping between the full exterior algebras of the tangent spaces to M, and we integrate with respect to the Lebesgue measure μ. The measure μ is not supposed to be invariant under g. If g is measure-preserving, then the same statement holds for k-vector fields; see [Koz2]. The topological entropy also gives a lower bound for the growth of the magnetic field in any L^q-norm ($q \ge \dim(M) - 2$), even in the case of finite smoothness of the diffeomorphism; see [K-Y].

Theorem 5.2 provides a necessary and sufficient condition for the existence of the exponential growth in a nondissipative L^1-dynamo. In order to be a dynamo, the velocity field v has to have nonzero entropy; i.e., roughly speaking, to admit some chaos. Positive topological entropy is often related to the presence of horseshoes, and they essentially exhaust all the entropy for two-dimensional systems ([Kat2]; see the discussion in [K-Y]). For other L^q-norms ($q > 1$), the topological entropy

gives a lower bound for the growth rate of an appropriate magnetic field [Koz2, K-Y].

The spectrum of the nondissipative kinematic dynamo operator for a continuous velocity field on a compact Riemannian manifold without boundary is described in [CLMS] (see also [LL]).

5.C. Upper bounds for dissipative L^1-dynamos

Klapper and Young [K-Y] proved that the same necessary condition is valid for dissipative (realistic) dynamos: If the topological entropy of the field vanishes $(h_{\text{top}}(v) = 0)$, then the field v cannot be a fast dynamo (in other words, the increment $\lambda(\eta)$ goes to zero as the magnetic diffusivity η tends to zero). Such a bound was proposed by Finn and Ott in 1988, and the proof was announced in 1992 by M.M. Vishik. The result and proof in [K-Y] is given in the more general form of finite smoothness of the magnetic and velocity fields:

Theorem 5.3 [K-Y]. *Let v and B_0 be divergence-free vector fields supported on a compact domain $M \subset \mathbb{R}^n$. Assume that v is of class C^{k+1} and B_0 is of class C^k for some $k \geq 2$. Let $B_\eta(t)$ be the solution of the dynamo equation (1.1) with the initial condition $B_\eta(0) = B_0$. Then*

$$\limsup_{\eta \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln \int_M \|B_\eta(n)\| \mu \leq h_{\text{top}}(v) + \frac{r(g)}{k},$$

where $g := g_v^1$ is the time 1 map of the flow defined by the field v, $B_\eta(n)$ is the value of $B_\eta(t)$ at the moments $t = n$, and

$$r(g) := \lim_{n \to \infty} \frac{1}{n} \ln \left(\max_{x \in M} \|\frac{\partial (g^n)}{\partial x}(x)\| \right).$$

(Here $\frac{\partial (g^n)}{\partial x}$ is the Jacobian matrix of the map g^n.) This upper bound is also valid for the vanishing magnetic diffusivity $\eta = 0$.

In the case of an idealized nondissipative dynamo $\eta = 0$ and a smooth vector field v ($k = \infty$) this theorem reduces to Theorem 5.2.

A variety of questions related to the kinematic dynamo are discussed in the recent book [ChG], which deals particularly with the fast dynamo problem, as well as in the books and surveys [Mof3, K-R, R-S, Chi2, ZRS, Z-R].

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Chapter VI

Dynamical Systems with Hydrodynamical Background

This chapter is a survey of several relevant systems to which the group-theoretic scheme of the preceding chapters or its modifications can be applied. The choice of topics for this chapter was intended to show different (but nevertheless, "hydrodynamical") features of a variety of dynamical systems and to emphasize suggestive points for further study and future results.

§1. The Korteweg–de Vries equation as an Euler equation

In Chapter I we discussed the common Eulerian nature of the equations of a three-dimensional rigid body and of an ideal incompressible fluid. The first equation is related to the Lie group $SO(3)$, while the second is related to the huge infinite-dimensional Lie group $S\operatorname{Diff}(M)$ of volume-preserving diffeomorphisms of M.

In this section we shall deal with an intermediate case: the Lie group of all diffeomorphisms of a one-dimensional object, the circle, or rather, with the one-dimensional extension of this group called the Virasoro group. In a sense it is the "simplest possible" example of an infinite-dimensional Lie group. It turns out that the corresponding Euler equation for the geodesic flow on the Virasoro group is well known in mathematical physics as the Korteweg–de Vries equation. This equation is widely regarded as a canonical example of an integrable Hamiltonian system with an infinite number of degrees of freedom.

1.A. Virasoro algebra

The Virasoro algebra is an object that is only one dimension larger than the Lie algebra $\operatorname{Vect}(S^1)$ of all smooth vector fields on the circle S^1 (in the physics literature, these vector fields are usually assumed to be trigonometric polynomials).

Definition 1.1. The *Virasoro algebra* (denoted by *vir*) is the vector space $\text{Vect}(S^1) \oplus \mathbb{R}$ equipped with the following commutation operation:

$$[(f(x)\frac{\partial}{\partial x}, a), (g(x)\frac{\partial}{\partial x}, b)]$$
$$= \left((f'(x)g(x) - f(x)g'(x))\frac{\partial}{\partial x}, \int_{S^1} f'(x)g''(x)\, dx\right),$$

for any two elements $(f(x)\partial/\partial x, a)$ and $(g(x)\partial/\partial x, b)$ in *vir*.

The commutator is a pair consisting of a vector field and a number. The vector field is minus the Poisson bracket of the two given vector fields on the circle: $\{f\,\partial/\partial x,\, g\,\partial/\partial x\} = (fg' - f'g)\,\partial/\partial x$. The bilinear skew-symmetric expression $c(f, g) := \int_{S^1} f'(x)g''(x)dx$ is called the *Gelfand–Fuchs 2-cocycle*; see [GFu].

Definition 1.2. A *real-valued two-cocycle* on an arbitrary Lie algebra $\mathfrak{g}$ is a bilinear skew-symmetric form $c(\cdot, \cdot)$ on the algebra satisfying the following identity:

$$\sum_{(f,g,h)} c([f, g], h) = 0, \qquad \text{for any three elements} \quad f, g, h \in \mathfrak{g},$$

where the sum is considered over the three cyclic permutations of the elements (f, g, h).

The cocycle identity means that the extended space $\hat{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{R}$ with the commutator defined by

$$(1.1) \qquad\qquad [(f, a),\, (g, b)] = ([f, g],\, c(f, g))$$

obeys the Jacobi identity of a Lie algebra. One can define $c(f, g)$ in (1.1) by setting $c(f, g) = 0$ for all pairs f, g and get a *trivial extension* of the Lie algebra $\mathfrak{g}$. An extension of the algebra $\mathfrak{g}$ is called *nontrivial* (or the corresponding 2-cocycle *is not a 2-coboundary*) if it cannot be reduced to the extension by means of the zero cocycle via a linear change of coordinates in $\hat{\mathfrak{g}}$. We discuss cocycles on Lie algebras, as well as the geometric meaning of the Gelfand–Fuchs cocycle, in more detail in Section 1.D.

The Virasoro algebra is the unique nontrivial one-dimensional central extension of the Lie algebra $\text{Vect}(S^1)$ of vector fields on the circle. There exists a Virasoro group whose Lie algebra is the Virasoro algebra *vir*; see, e.g., [Ner1].

Definition 1.3. The *Virasoro* (or *Virasoro–Bott*) *group* is the set of pairs $(\varphi(x), a) \in \text{Diff}(S^1) \oplus \mathbb{R}$ with the multiplication law

$$(\varphi(x),\, a) \circ (\psi(x),\, b) = \left(\varphi(\psi(x)),\, a + b + \int_{S^1} \log(\varphi \circ \psi(x))'\, d\log \psi'(x)\right).$$

Applying the general constructions of Chapter I to the Virasoro group, we equip this group with a (right-invariant) Riemannian metric. For this purpose we fix the

energy-like quadratic form in the Lie algebra vir; i.e., on the tangent space to the group identity:

$$H(f(x)\frac{\partial}{\partial x}, a) = \frac{1}{2}\left(\int_{S^1} f^2(x)\, dx + a^2\right).$$

Consider the corresponding Euler equation; i.e., the equation of the geodesic flow generated by this metric on the Virasoro group.

Definition 1.4. The *Korteweg–de Vries* (KdV) equation on the circle is the evolution equation

$$\partial_t u + uu' + u''' = 0$$

on a time-dependent function u on S^1, where $' = \partial/\partial x$ and $\partial_t = \partial/\partial t$; see [KdV].

Theorem 1.5 [OK1]. *The Euler equation corresponding to the geodesic flow (for the above right-invariant metric) on the Virasoro group is a one-parameter family of KdV equations.*

Proof. The equation for the geodesic flow on the Virasoro group corresponds to the Hamiltonian equation on the dual Virasoro algebra vir^*, with the linear Lie–Poisson bracket and the Hamiltonian function $-H$.

The space vir^* can be identified with the set of pairs

$$\left\{(u(x)(dx)^2, c) \mid u(x) \text{ is a smooth function on } S^1,\ c \in \mathbb{R}\right\}.$$

Indeed, it is natural to contract the quadratic differentials $u(x)(dx)^2$ with vector fields on the circle, while the constants are to be paired between themselves:

$$\langle(v(x)\frac{\partial}{\partial x}, a), (u(x)(dx)^2, c)\rangle = \int_{S^1} v(x) \cdot u(x)\, dx + a \cdot c.$$

The coadjoint action of a Lie algebra element $(f\,\partial/\partial x, a) \in vir$ on an element $(u(x)(dx)^2, c)$ of the dual space vir^* is

(1.2) $$\mathrm{ad}^*_{(f\,\partial/\partial x, a)}(u(dx)^2, c) = (2f'u + fu' + cf''', 0),$$

where $'$ stands for the x-derivative. It is obtained from the identity

$$\langle[(f\frac{\partial}{\partial x}, a), (g\frac{\partial}{\partial x}, b)], (u(dx)^2, c)\rangle = \langle(g\frac{\partial}{\partial x}, b), \mathrm{ad}^*_{(f\frac{\partial}{\partial x}, a)}(u(dx)^2, c)\rangle,$$

which holds for every pair $(g\frac{\partial}{\partial x}, b) \in vir$.

The quadratic energy functional H on the Virasoro algebra determines the "tautological" inertia operator $A : vir \to vir^*$, which sends a pair $(u(x)\partial/\partial x, c) \in vir$ to $(u(x)(dx)^2, c) \in vir^*$.

In particular, it defines the quadratic Hamiltonian on the dual space vir^*,

$$H(u(dx)^2, c) = \frac{1}{2}(\int u^2\, dx + c^2)$$

$$= \frac{1}{2}\langle(u\frac{\partial}{\partial x}, c), (u(dx)^2, c)\rangle = \frac{1}{2}\langle(u\frac{\partial}{\partial x}, c), A(u\frac{\partial}{\partial x}, c)\rangle.$$

The corresponding Euler equation for the right-invariant metric on the group (according to the general formula (I.6.4), Theorem I.6.15) is given by

$$\frac{\partial}{\partial t}(u(dx)^2, c) = -\operatorname{ad}^*_{A^{-1}(u(dx)^2,c)}(u(dx)^2, c).$$

Making use of the explicit formula for the coadjoint action (1.2) with

$$(f\,\partial/\partial x, a) = A^{-1}(u(dx)^2, c) = (u\,\partial/\partial x, c),$$

we get the required Euler equation:

$$\begin{cases} \partial_t u = -2u'u - uu' - cu''' = -3uu' - cu''', \\ \partial_t c = 0. \end{cases}$$

The coefficient c is preserved in time, and the function u satisfies the KdV equation (with different coefficients). □

Remark 1.6. Without the central extension, the Euler equation on the group of diffeomorphisms of the circle has the form

$$\partial_t u = -3uu'.$$

(called a nonviscous Burgers equation). Rescaling time, this equation can be reduced to the equation on the velocity distribution u of freely moving noninteracting particles on the circle. It develops completely different properties as compared to the KdV equation (see, e.g., [Arn15]).

If $u(x, t)$ is the velocity of a particle x at moment t, then the substantial derivative of u is equal to zero: $\partial_t u + uu' = 0$. In Fig. 71 one sees a typical perestroika of the velocity field u in time. Since every particle keeps its own velocity, fast particles pass by slow ones. Every point of inflection in the initial velocity profile $u(x, 0)$ generates a shock wave.
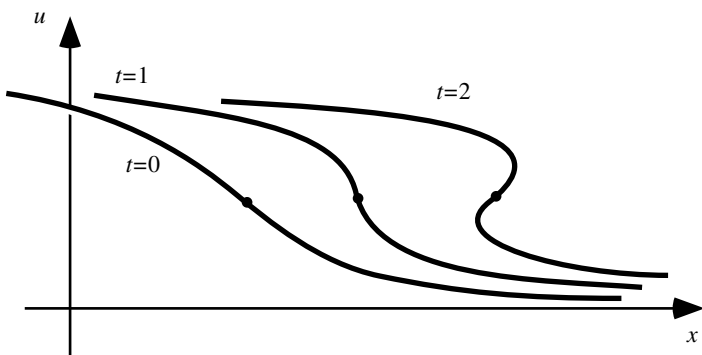
FIGURE 71. The shock wave generated by freely moving noninteracting particles.

Thus, in finite time, solutions of the corresponding Euler equation define a multivalued, rather than univalued, vector field of the circle. In other words, the

geodesic flow on the group $\mathrm{Diff}(S^1)$, with respect to the right-invariant metric generated by the quadratic form $\int u^2 dx$ on the Lie algebra, is incomplete.

Note also that in the case of the Burgers equation with small viscosity, shock waves appear as well. Initial series of typical bifurcations of shock waves were described in [Bog] (see also [SAF, Si2]). Typical singularities of projections of the solutions on the plane of independent variables for 2×2 quasilinear systems are classified in [Ra].

On the other hand, the solutions of the KdV equation exist and remain smooth for all t, and do not develop shock waves. In the interpretation of the KdV as the shallow water equation, the parameter c measures dispersion of the medium.

Remark 1.7. Differential geometry of the Virasoro group with respect to the above right-invariant metric is discussed in [Mis3]. In particular, the sectional curvatures in the two-dimensional directions containing the central direction are nonnegative (cf. Remark IV.2.4). For the relation between the geometry of the KdV equation and the Kähler geometry of the Virasoro coadjoint orbits see [Seg, STZ].

One can extend this Eulerian viewpoint to the super-KdV equation (introduced in [Kup]) and describe the latter as the equation of geodesics on the super-analogues of the Virasoro group, corresponding to the Neveu–Schwarz and Ramond super-algebras; see [OK1]. Another elaboration of this viewpoint is the passage from the L^2-metric on the Virasoro algebra to another one, say, the H^1-metric:

$$\|(f(x)\frac{\partial}{\partial x},\, a)\|^2_{H^1} = \frac{1}{2}\left(\int_{S^1} f^2(x)\, dx + \int_{S^1} (f'(x))^2\, dx + a^2 \right).$$

Proposition 1.8 [Mis4]. *The H^1-metric on the central extension of the Lie algebra* $\mathrm{Vect}(S^1)$ *of vector fields on the circle given by the (trivial) 2-cocycle*

$$c(f, g) := \int_{S^1} f'(x)g(x)\, dx$$

generates the shallow water equation

$$\begin{cases} \partial_t u - \partial_t u'' = uu''' + 2u'u'' - 3uu' - cu', \\ \partial_t c = 0 \end{cases}$$

(introduced in [C-H]). Here the prime ' stands for $\frac{\partial}{\partial x}$, and ∂_t denotes $\frac{\partial}{\partial t}$.

1.B. The translation argument principle and integrability of the high-dimensional rigid body

Definitions 1.9. A function F on a symplectic manifold is a *first integral of a Hamiltonian system* with Hamiltonian H if and only if the Poisson bracket of H

with F is equal to zero. Functions whose Poisson bracket is equal to zero are said to be *in involution* with respect to this bracket.

A Hamiltonian system on a symplectic $2n$-dimensional manifold M^{2n} is called *completely integrable* if it has n integrals in involution that are functionally independent almost everywhere on M^{2n}.

A theorem attributed to Liouville states that connected components of non-critical common level sets of n first integrals on a compact manifold are the n-dimensional tori. The Hamiltonian system defines a quasiperiodic motion $\dot{\phi} = \text{const}$ in appropriate angular coordinates $\phi = (\phi_1, \ldots, \phi_n)$ on each of the tori; see [Arn16].

Example 1.10. Every Hamiltonian system with one degree of freedom is completely integrable, since it always possesses one first integral, the Hamiltonian function itself.

In particular, the Euler equation of a three-dimensional rigid body is a completely integrable Hamiltonian system on the coadjoint orbits of the Lie group $SO(3)$. These orbits are the two-dimensional spheres centered at the origin and the origin itself, while the Hamiltonian function is given by the kinetic energy of the system.

Example 1.11. A consideration of dimensions is not enough to argue the complete integrability of the equation of an n-dimensional rigid body for $n > 3$. Free motions of a body with a fixed point are described by the geodesic flow on the group $SO(n)$ of all rotations of Euclidean space $\mathbb{R}^n$.

The group $SO(n)$ is equipped with a particular left-invariant Riemannian metric defined by the inertia quadratic form in the body's internal coordinates. On the Lie algebra $\mathfrak{so}(n)$ of skew-symmetric $n \times n$ matrices this quadratic form is given by $-\operatorname{tr}(\omega D\omega)$, where

$$\omega \in \mathfrak{so}(n), \quad D = \operatorname{diag}(d_1, \ldots, d_n), \quad d_k = \frac{1}{2} \int \rho(x) x_k^2 \, d^n x,$$

and where $\rho(x)$ is the density of the body at the point $x = (x_1, \ldots, x_n)$. The inertia operator $A : \mathfrak{so}(n) \to \mathfrak{so}(n)^*$ defining this quadratic form sends a matrix ω to the matrix $A(\omega) = D\omega + \omega D$.

Remark 1.12. For $n = 3$ this formula implies the triangle inequality for the principal momenta d_k. Operators satisfying these inequalities form an open set in the space of symmetric 3×3 matrices. In higher dimension ($n > 3$) the symmetric matrices representing the inertia operators of the rigid bodies are very special. They form a variety of dimension n in the $n(n-1)/2$-dimensional space of equivalence classes of symmetric matrices on the Lie algebra.

Theorem 1.13 ([Mish] **for** $n = 4$, [Man] **for all** n). *The Euler equation $\dot{m} = \text{ad}^*_\omega m$ of an n-dimensional rigid body, where $\omega = A^{-1}m$ and the inertia operator A is defined above, is completely integrable. The functions*

$$(1.3) \qquad\qquad H_{\lambda,\nu} = \det(m + \lambda D^2 + \nu E)$$

on the dual space $\mathfrak{so}(n)^*$ provide a complete family of integrals in involution.

The involutivity of the quantities $H_{\lambda,\nu}$ can be proved by the method of Poisson pairs and translation of the argument, which we discuss below (see [Man]). Note that the physically meaningful inertia operators $A(\omega) = D\omega + \omega D$ (i.e., those with entries $a_{ij} = d_i + d_j$) form a very special subset in the space of all symmetric operators $A : \mathfrak{so}(n) \to \mathfrak{so}(n)^*$. According to Manakov, a sufficient condition for integrability is that

$$a_{ij} = \frac{p_i - p_j}{q_i - q_j}$$

(which for $p_i = q_i^2$ becomes the physical case above). The limit $n \to \infty$ of the integrable cases on $SO(n)$ was considered in [War].

The geodesic flow on the group $SO(n)$ equipped with an arbitrary left-invariant Riemannian metric is, in general, nonintegrable.

Usually, integrability of an infinite-dimensional Hamiltonian system is related to the existence of two independent Poisson structures forming a so-called Poisson pair, such that the system is Hamiltonian with respect to both structures.

Definitions 1.14. Assume that a manifold M is equipped with two Poisson structures $\{\cdots\}_0$ and $\{\cdots\}_1$. They are said to *form a Poisson pair* (or to *be compatible*) if all of their linear combinations $\lambda\{\cdots\}_0 + \nu\{\cdots\}_1$ are also Poisson structures.

A dynamical system $\dot{x} = v(x)$ on M is called *bi-Hamiltonian* if the vector field v is Hamiltonian with respect to both structures $\{\cdots\}_0$ and $\{\cdots\}_1$.

Remark 1.15. The condition on $\{\cdots\}_0$ and $\{\cdots\}_1$ to form a Poisson pair is equivalent to the identity

$$(1.4) \qquad\qquad \sum_{(f,g,h)} \{\{f,g\}_0, h\}_1 + \{\{f,g\}_1, h\}_0 = 0$$

for any triple of smooth functions f, g, h on M, where the sum is taken over all three cyclic permutations of the triple.

In the next theorem we assume, for the sake of simplicity, that M is simply connected and that the Poisson structures $\{\cdots\}_0$ and $\{\cdots\}_1$ are everywhere nondegenerate; i.e., they are inverses of some symplectic structures on M.

Theorem 1.16 [GDo]. *Let v be a bi-Hamiltonian vector field with respect to the structures of a Poisson pair $\{\cdots\}_0$, $\{\cdots\}_1$. Then there exists a sequence of smooth functions H_k, $k = 0, 1, \ldots$, on M such that*

(1) H_0 is a Hamiltonian of the field $v_0 := v$ with respect to the structure $\{\cdots\}_0$;

(2) the field v_k of the 0-Hamiltonian H_k coincides with the field of the 1-Hamiltonian H_{k+1};

(3) the functions H_k, $k = 0, 1, \ldots$, are in involution with respect to both Poisson brackets.

The algorithm for generating the Hamiltonians H_k is called the *Lenard scheme* and is shown in Fig. 72.
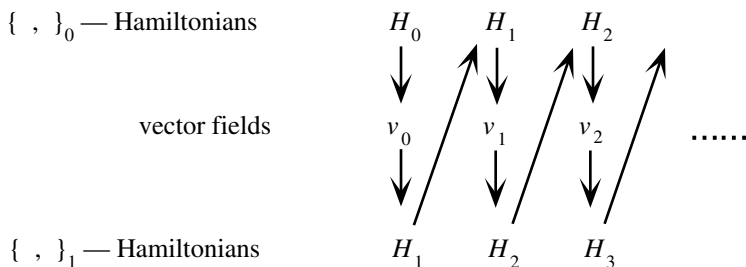
FIGURE 72.   Generation of a sequence of the Hamiltonians H_k for a bi-Hamiltonian vector field.

Although this theorem is formulated and proven for the case of nondegenerate brackets only, the procedure is usually applied in a more general context. Namely, let the field $v_0 := v$ be Hamiltonian with Hamiltonian functions H_0 and H_1 relative to the structures $\{\cdots\}_0$ and $\{\cdots\}_1$, respectively. Consider the function H_1 as the Hamiltonian with respect to the bracket $\{\cdots\}_0$ and generate the next Hamiltonian field v_1. One readily shows that the field v_1 *preserves* the Poisson bracket $\{\cdots\}_0$, provided that the two brackets form a Poisson pair (a formal application of the identity (1.4)). However, this does not imply, in general, that the field v_1 is *Hamiltonian* with respect to the bracket $\{\cdots\}_0$. (Example: The vertical field $\partial/\partial z$ preserves the Poisson structure in $\mathbb{R}^3_{x,y,z}$ given by the bivector field $\partial/\partial x \wedge \partial/\partial y$, but it is not defined by any Hamiltonian function. Every Hamiltonian field for this structure would be horizontal.) If we are lucky, and the field v_1 is indeed Hamiltonian, we continue the process to the next step, and so on.

To apply the technique of Poisson pairs in the Lie-algebraic situation, recall that on the dual space $\mathfrak{g}^*$ to any Lie algebra $\mathfrak{g}$ there exists a natural linear Lie–Poisson structure (see Section I.6)

$$\{f, g\}(m) := \langle [df, dg], m \rangle$$

for any two smooth functions f, g on $\mathfrak{g}^*$, and $m \in \mathfrak{g}^*$. In other words, the Poisson bracket of two linear functions on $\mathfrak{g}^*$ is equal to their commutator in the Lie algebra $\mathfrak{g}$ itself. The symplectic leaves of this Poisson structure are the coadjoint orbits

of the group action on $\mathfrak{g}^*$, while the Casimir functions are invariant under the coadjoint action. The following method of constructing functions in involution on the orbits is called the *method of translation of the argument*, and originally appeared in Manakov's paper [Man] to describe the integrable cases of higher-dimensional rigid bodies (see further generalizations in, e.g., [A-G, T-F]).

Fix a point m_0 in the dual space to a Lie algebra. One can associate to this element a new Poisson bracket on $\mathfrak{g}^*$.

Definition 1.17. The *constant Poisson bracket associated to a point* $m_0 \in \mathfrak{g}^*$ is the bracket $\{\cdots\}_0$ on the dual space $\mathfrak{g}^*$ defined by

$$\{f, g\}_0(m) := \langle [df, dg], m_0 \rangle$$

for any two smooth functions f, g on the dual space, and any $m \in \mathfrak{g}^*$. The differentials df, dg of the functions f, g are taken at a current point m, and, as above, are regarded as elements of the Lie algebra itself.

The brackets $\{\cdots\}$ and $\{\cdots\}_0$ coincide at the point m_0 itself. Moreover, the bivector defining the constant bracket $\{\cdots\}_0$ does not depend on the current point. The symplectic leaves of the bracket are the tangent plane to the group coadjoint orbit at the point m_0, as well as all the planes in $\mathfrak{g}^*$ parallel to this tangent plane (Fig. 73).
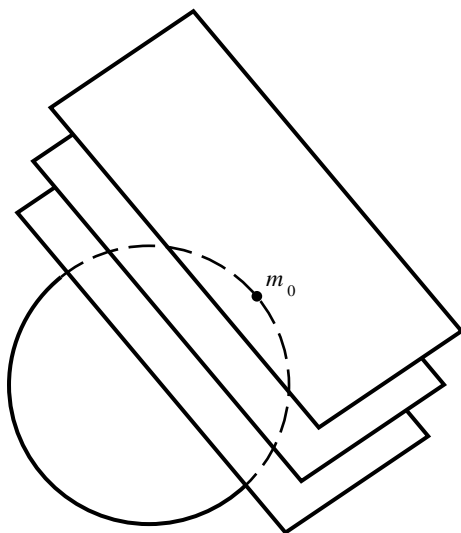
FIGURE 73. Symplectic leaves of the constant bracket are the planes parallel to the tangent plane to the coadjoint orbit at m_0.

Proposition 1.18. *The brackets* $\{\cdots\}$ *and* $\{\cdots\}_0$ *form a Poisson pair for every fixed point* m_0.

Proof. The linear combination $\{\cdots\}_\lambda := \{\cdots\} + \lambda\{\cdots\}_0$ is a Poisson bracket, being the linear Lie–Poisson structure $\{\cdots\}$ translated from the origin to the point $-\lambda m_0$. □

Corollary 1.19. *Let* $f, g : \mathfrak{g}^* \to \mathbb{R}$ *be invariants of the group coadjoint action, and let* $m_0 \in \mathfrak{g}^*$. *Then the functions* $f(m + \lambda m_0)$, $g(m + v m_0)$ *of the point* $m \in \mathfrak{g}^*$ *are in involution for any* $\lambda, v \in \mathbb{R}$ *on each coadjoint orbit.*

Proof. This is an immediate consequence of the fact that Casimir functions for all linear combinations of compatible Poisson brackets are in involution with each other. The latter holds by virtue of the definition of a Poisson pair. □

We leave it to the reader to adjust this corollary to produce the family (1.3) of first integrals providing the integrability of the higher-dimensional rigid body (see [Man]). Below, we show how this scheme works for the KdV equation.

Remark 1.20. A Hamiltonian function f and the Poisson structure $\{\cdots\}_0$ generate the following Hamiltonian vector field on the dual space $\mathfrak{g}^*$:

$$(1.5) \qquad\qquad v(m) = \mathrm{ad}^*_{df}\, m_0,$$

where the differential df is taken at the point m. Indeed, for an arbitrary function g one has

$$\{f, g\}_0(m) = \langle \mathrm{ad}_{df}(dg), m_0 \rangle = \langle (dg), \mathrm{ad}^*_{df}\, m_0 \rangle,$$

and the latter pairing is the Lie derivative $L_v g$ (at the point $m \in \mathfrak{g}^*$) of the function g along the field v defined by (1.5). Hence, this vector field v is Hamiltonian with Hamiltonian function f.

1.C. Integrability of the KdV equation

The existence of an infinite number of conserved charges for the flow determined by the KdV equation was discovered in the late 1960s, and in a sense this discovery launched the modern theory of infinite-dimensional integrable Hamiltonian systems (see [Ma, Miu] for an intriguing historical survey).

The first of the KdV conservation laws were found via calculations with undetermined coefficients, but this method stopped at the 9th invariant. Miura describes in [Miu] how, in the summer of 1966, a rumor circulated that there were exactly 9 conservation laws in this case. Miura spent a week of his summer vacation and succeeded in finding the 10th one. Later code was written computing the 11th law. After that the specialists were convinced that there should be an infinite series of conservation laws.

In this section, we shall see how these laws can be extracted via the recursive Lenard scheme, or equivalently, via Manakov's method of the translation of argument from the preceding section.

The KdV equation is an example of a bi-Hamiltonian system. First, as we discussed in Section 1.A, it is Hamiltonian on the dual space $vir^* = \{(u(dx)^2, c)\}$ of the Virasoro algebra with the quadratic Hamiltonian function

$$-H(u(dx)^2, c) = -\frac{1}{2}\left(\int u^2\, dx + c^2\right)$$

relative to the linear Poisson structure. This Poisson structure is called the *second KdV Hamiltonian structure* and is sometimes referred to as the Magri bracket; see [Mag].

Moreover, one can specify a point in the space vir^* such that the KdV equation will also be Hamiltonian with respect to the constant Poisson structure associated to this point. Namely, let the pair $(u_0(x)(dx)^2, c_0)$ consist of the function $u_0(x) \equiv 1/2$ and $c_0 = 0$.

Definition 1.21. Let F be a function on the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ and $m \in \mathfrak{g}^*$. In the case of an infinite-dimensional space $\mathfrak{g}^*$, the differential $dF|_m$ (regarded as a vector of the Lie algebra itself) is called the *variational derivative* $\frac{\delta F}{\delta m}$, and it is defined by the relation

$$\frac{d}{d\epsilon}F(m + \epsilon w)\Big|_{\epsilon=0} = \langle\frac{\delta F}{\delta m},\, w\rangle.$$

For instance, in the case of the Virasoro algebra, a functional F is defined on the set of pairs $(u(x)(dx)^2, c)$. The variational derivative

$$\left(\left(\frac{\delta F}{\delta u}\right)\frac{\partial}{\partial x},\, \frac{\delta F}{\delta c}\right)$$

is the pair consisting of a vector field and a number such that

$$\frac{d}{d\epsilon}F((u + \epsilon w)(dx)^2,\, c + \epsilon b)\Big|_{\epsilon=0} = \langle\left(\left(\frac{\delta F}{\delta u}\right)\frac{\partial}{\partial x},\, \frac{\delta F}{\delta c}\right),\, (w(dx)^2, b)\rangle$$

$$= \int\left(\frac{\delta F}{\delta u}(x)\cdot w(x)\right)dx + \frac{\delta F}{\delta c}\cdot b.$$

(To specify the class of functionals, one usually considers differential polynomials on vir^*; i.e., integrals of polynomials in u and in its derivatives; see [GDo]).

Proposition 1.22. *(i) The Poisson structure $\{\cdots\}_0$ associated to the point*

$$\left(\frac{1}{2}(dx)^2,\, 0\right) \in vir^*$$

sends every Hamiltonian function F on the dual space vir^* to the Hamiltonian vector field on vir^* whose value at a point $(u(dx)^2, c)$ is the pair

$$\left(\left(\frac{\delta F}{\delta u}\right)'(x)(dx)^2,\, 0\right).$$

(ii) The Korteweg–de Vries equation is Hamiltonian with respect to the constant Poisson structure $\{\cdots\}_0$ with the Hamiltonian function

$$(1.6) \qquad Q(u(dx)^2, c) = \frac{1}{2} \int_{S^1} \left(-u^3(x) + c(u')^2(x) \right) \, dx.$$

The Poisson structure $\{\cdots\}_0$ is called the *first KdV Hamiltonian structure*; see, e.g., [LeM]. The Hamiltonians $H_2 = H$ and $H_3 = Q$ of the KdV equation with respect to the Poisson pair $\{\cdots\}$ and $\{\cdots\}_0$ start the series of conservation laws generated by the Lenard iteration scheme. One readily shows that at each step the Hamiltonian functional H_k is a differential polynomial of order k in $u(x)$. Usually, this series of first integrals for the KdV starts with the Hamiltonian function $H_1(u) := \int u(x) \, dx$.

Proof. Item (i) is a straightforward application of the notion of variational derivative to formula (1.5). Indeed, one obtains the Hamiltonian vector field for a functional F on the space vir^* by freezing the values of $u(x)$ and c as $u_0(x) \equiv 1/2$ and $c_0 = 0$ in (1.2):

$$\mathrm{ad}^*_{(f\, \partial/\partial x, a)}(u_0(dx)^2, c_0) = \left((2f'u_0 + fu_0' + c_0 f''')(dx)^2, \, 0 \right) = \left(f'(dx)^2, \, 0 \right),$$

where $f := \left(\frac{\delta F}{\delta u} \right)$, and $a := \left(\frac{\delta F}{\delta c} \right)$.

(ii) The variational derivative of the functional Q given by (1.6) is

$$(1.7) \qquad\qquad\qquad \left(\frac{\delta Q}{\delta u} \right) = -\frac{3}{2} u^2 - cu''.$$

Indeed, this follows from the equality

$$\frac{d}{d\epsilon} \frac{1}{2} \int_{S^1} \left(-(u + \epsilon w)^3 + c \left((u + \epsilon w)' \right)^2 \right) dx = -\int_{S^1} \left(\frac{3}{2} u^2 + cu'' \right) \cdot w \, dx.$$

Then, substituting the variational derivative $f = \delta Q/\delta u$ from (1.7) into (1.2), we get the following Hamiltonian vector field on the dual space vir^*:

$$\begin{cases} \partial_t u = f' = -(\frac{3}{2}u^2 + cu'')' = -3uu' - cu''', \\ \partial_t c = 0, \end{cases}$$

that is, the KdV equation. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 1.23. The KdV flow is tangent to the coadjoint orbits of the Virasoro algebra (as is the flow of every Euler equation on the dual space to any Lie algebra). Note that none of the above first integrals of the KdV equation are invariants of the Virasoro coadjoint action, and therefore their meaning is completely different from the Casimir functions of two-dimensional hydrodynamics (cf. Remark I.9.8). The description of the Virasoro orbits (or Casimir functions), besides being evident information on the behavior of KdV solutions, is an interesting question in its own right.

The classification problem for the Virasoro coadjoint orbits is also known as the classification of Hill's operators

$$\left\{ \frac{d^2}{dx^2} + u(x) \mid u \in C^\infty(S^1) \right\},$$

or of projective structures on the circle, and it has been solved independently in different terms and at different times (see [Kui, LPa, Seg, Ki2]). The orbits are enumerated by one discrete parameter and one continuous parameter. Generalization of this problem to the classification of symplectic leaves of the so-called Gelfand–Dickey brackets, which are certain natural Poisson brackets on differential operators of higher order on the circle, as well as the relation of this problem to enumeration of homotopy types of nonflattening curves on spheres, is given in [OK2] (see also [KhS, Sha, E-K] for relevant problems).

The Lenard scheme generates a series of Hamiltonian equations called the KdV hierarchy. A similar construction exists for higher KdV hierarchies, which are Hamiltonian flows on coefficients of differential operators of higher order on the circle; see [Adl, GDi, SeW, PrS].

1.D. Digression on Lie algebra cohomology and the Gelfand–Fuchs cocycle

The theory of Lie algebra cohomology is an algebraic generalization of the following geometric construction from Lie group theory.

Let G be a compact connected simply connected Lie group equipped with a two-sided invariant metric (a typical example: the group of unit quaternions $SU(2) \approx S^3$ or the group $SU(n)$ of unitary matrices with unit determinant).

One can calculate the cohomology groups and, furthermore, their exterior algebra for the group G as follows.

Theorem 1.24. *The exterior algebra of two-sided invariant differential forms on G is isomorphic to the cohomology exterior algebra of the manifold G. The isomorphism is defined by assigning to each differential form its cohomology class.*

The proof is based on two facts: (i) every two-sided invariant form is closed (see (1.8) below); (ii) every closed 2-form is cohomologous to a two-sided invariant form, namely, to the average value of all its shifts.

This classical theorem reduces all calculations to a purely algebraic consideration in terms of the commutator of the Lie algebra. Indeed, any one-sided invariant form is determined by its value on the Lie algebra. The exterior differential of this form is also an invariant form, and hence it is also determined by its value on the Lie algebra.

Given an invariant 1-form ω, its differential is a 2-form defined at the identity by the following *Maurer–Cartan formula*:

$$(d\omega)(\xi, \eta) = \mp\omega([\xi, \eta])$$

(the sign is defined by whether the form is left- or right-invariant). This formula allows one to write algebraically the closedness condition (and to verify that it coincides with the condition of two-sided invariance). More generally, one has the following

Theorem 1.25. *Given a one-sided invariant n-form ω whose value on the Lie algebra $\mathfrak{g}$ is $\omega(\xi_1, \ldots, \xi_n)$, $\xi_i \in \mathfrak{g}$, its exterior differential is the invariant $(n+1)$-form $d\omega$ whose value on the Lie algebra is*

(1.8)
$$d\omega(\xi_0, \ldots, \xi_n) = \pm \sum_{0 \leq i < j \leq n} (-1)^{i+j} \omega([\xi_i, \xi_j], \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_n),$$

where the sign is determined by whether the form ω is left- or right-invariant, and the hat $\hat{\ }$ means that the corresponding vector is missing.

Example 1.26. For a 1-form ω we have $\mp(d\omega)(\xi, \eta) = \omega([\xi, \eta])$. The differential of a 2-form ω is given by the formula

$$\mp d\omega(\xi, \eta, \zeta) = \omega([\xi, \eta], \zeta) + \omega([\eta, \zeta], \xi) + \omega([\zeta, \xi], \eta).$$

The algebraic generalization mentioned above, which allows one to avoid calculations on the Lie group, proceeds as follows.

Definition 1.27. The *cohomology complex of a Lie algebra* $\mathfrak{g}$ is the complex

$$\Omega^0 \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \Omega^2 \xrightarrow{d_2} \cdots,$$

where Ω^n is a vector space of exterior n-forms on the Lie algebra $\mathfrak{g}$, and the differential d_n is given by formula (1.8).

The nth-cohomology group (or space) of the Lie algebra $\mathfrak{g}$ is the vector space

$$\frac{\mathrm{Ker}\, d_n : \Omega^n \to \Omega^{n+1}}{\mathrm{Im}\, d_{n-1} : \Omega^{n-1} \to \Omega^n},$$

that is, the quotient of the space of all closed n-forms over the subspace of all exact forms. The elements of the space $\mathrm{Ker}\, d_n : \Omega^n \to \Omega^{n+1}$ are called n-*cocycles*, while the elements of the subspace $\mathrm{Im}\, d_{n-1} : \Omega^{n-1} \to \Omega^n$ are n-*coboundaries*, or the cocycles cohomologous to zero.

Remark 1.28. The fact that $d_n d_{n-1} = 0$ readily follows from formula (1.8) and the Jacobi identity. Geometrically, it means that the boundary of any simplex boundary (say, for a triangle or a tetrahedron) is zero.

Example 1.29. Let $a \in \mathfrak{g}^*$ be any element of the dual space to the Lie algebra $\mathfrak{g}$ and set

$$\omega(\xi, \eta) := a([\xi, \eta]).$$

This function is a 2-cocycle, and even a 2-coboundary, on $\mathfrak{g}$.

For instance, for the Lie algebra $\mathfrak{g} = \text{Vect}(S^1)$ of vector fields on the circle, and the point $a = (dx)^2/2 \in \text{Vect}^*$, we get the following 2-cocycle cohomologous to zero:

$$\omega\left(g(x)\frac{\partial}{\partial x}, h(x)\frac{\partial}{\partial x}\right) = \frac{1}{2}\int_{S^1}(g'h - gh')\,dx = \int_{S^1} g'h\,dx.$$

Remark 1.30. The 2-cocycle on $\text{Vect}(S^1)$ defining the Virasoro algebra is of a more subtle nature, and is related to the projective structures on the circle (we follow [Tab3] below).

Note that every 2-cocycle on a Lie algebra is a linear map from this Lie algebra to its dual. We construct a natural map from the Lie algebra $\text{Vect}(S^1)$ of vector fields on the circle to its dual, the space of quadratic differentials $\text{Vect}(S^1)^* = \{u(x)(dx)^2\}$, fixing first a projective structure on S^1.

Consider four points $x, x + t, x + 2t, x + 3t$ in an affine coordinate system, where t is very small. A diffeomorphism $f : S^1 \to S^1$ sends them to four points whose cross ratio is of order t^2 (not of order t!). The principal part of this cross ratio at the point x is (up to a constant factor) the *Schwarzian derivative* $S(x)$ of the diffeomorphism f:

$$S(x) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The corresponding quadratic differential $S(x)(dx)^2$ is independent of the (projective) choice of the coordinate x and measures the "nonprojectivity" of the map f. It is a cocycle of the diffeomorphism group of the circle with values in the quadratic differentials.

Now consider the Lie algebra of vector fields. Let f be a diffeomorphism of S^1 close to the identity, $f(x) = x + sv(x)$, where s is small, and let $v(x)d/dx$ be a vector field of the algebra $\text{Vect}(S^1)$. Then $S(x)$ is $sv'''(x) + O(s^2)$, where $'$ stands for d/dx. Neglecting higher-order terms, we get the desired mapping, which sends the field $v(x)\,d/dx$ to the quadratic differential $v'''(x)(dx)^2$.

For the angular coordinate q on the circle, the field $w(q)\partial/\partial q$ is sent to $(d^3w/dq^3 + dw/dq)(dq)^2$. This can be deduced with virtually no calculations from the description of the Lie algebra spanned by the generators of the projective group: $\partial/\partial q, (\cos q)\partial/\partial q, (\sin q)\partial/\partial q$ (accompanied by the change of variables normalizing the first coefficient).

We have now obtained the cocycle whose value on two vector fields $v(q)\partial/\partial q$ and $w(q)\partial/\partial q$ is given by the expression

$$\int_{S^1}(vw''' + vw')\,dq,$$

where $'$ is the derivative d/dq along the angular coordinate. The second term is cohomologous to zero, as we have seen above (see Example 1.29). Integrating by

parts the first monomial, we obtain the Gelfand–Fuchs cocycle. Thus the Gelfand–Fuchs cocycle (and hence, the Virasoro algebra) measures the deformation of the projective structure on $S^1 = \mathbb{R}P^1$ by diffeomorphisms.

§2. Equations of gas dynamics and compressible fluids

The evolution of a compressible fluid naturally extends the motion of an ideal incompressible fluid: Instead of the incompressibility condition, one assumes now that the pressure term of the Euler equation is determined by the intrinsic degrees of freedom of the fluid. Usually these internal parameters are the density and entropy of the fluid.

2.A. *Barotropic fluids and gas dynamics*

Barotropic fluids (or gas dynamics) are simplified models of compressible fluids in which the only intrinsic degree of freedom is the density of the fluid or of the gas.

Definition 2.1. A (compressible) fluid is *barotropic* (or *isentropic*) if the pressure term in the evolution equation is defined solely by the fluid's density. The fluid motion is described by the following system of equations:

$$(2.1) \qquad \begin{cases} \rho \, \dot{v} = -\rho \, (v, \nabla)v - \nabla h(\rho), \\ \dot{\rho} + \mathrm{div}(\rho v) = 0, \end{cases}$$

where v and ρ are respectively the velocity vector field and the density function of the fluid. The pressure function $h(\rho)$ depends on the physical properties of the fluid, and is assumed to be given. For instance, the equation of *gas dynamics* on a line corresponds to the choice $h(\rho) = \rho^\nu$ (for the motion of air $\nu \approx 1.4$).

Equations (2.1) make sense for an arbitrary Riemannian manifold M, provided that (v, ∇) stands for the covariant derivative along the field v (see Chapter I) and the divergence is taken with respect to the volume form induced by the metric. The first equation is similar to the Euler dynamics of an incompressible fluid, but the velocity field $v \in \mathrm{Vect}(M)$ is no longer divergence-free. The second equation is the continuity equation for the function ρ. Thus the phase space of the system consists of all pairs $\{(v, \rho) \mid v \in \mathrm{Vect}(M), \rho \in C^\infty(M)\}$.

The configuration space of the barotropic fluid on a manifold M is the group

$$P := \mathrm{Diff}\, M \ltimes C^\infty(M),$$

defined as the semidirect product of the group of *all* diffeomorphisms of M and the space $C^\infty(M)$ of all smooth functions on the manifold considered (see [HMRW] for a derivation of the equation via reduction in the Lagrangian representation).

Recall (cf. Section I.10 on the magnetic extension of a group) that the group structure on the *semidirect product P* is defined by the formula

$$(\varphi, a) \circ (\psi, b) = (\varphi \circ \psi, \psi_* a + b),$$

where $\psi_* a$ is the natural action of the diffeomorphism ψ on the function a: $\psi_* a = a(\psi^{-1}(x))$. The commutator in the corresponding Lie algebra

$$\mathfrak{p} = \mathrm{Vect}(M) \ltimes C^\infty(M)$$

is also defined via the semidirect product of the Lie algebras involved:

$$[(v, a), (w, b)] = ([v, w], L_w a - L_v b),$$

where $\varphi, \psi \in \mathrm{Diff}(M)$; $a, b \in C^\infty(M)$; $v, w \in \mathrm{Vect}(M)$, and $[v, w]$ denotes the commutator; i.e., minus the Poisson bracket, of the two vector fields on M ($[v, w] = -\{v, w\}$); see Section I.2.

Remark 2.2. The Lie algebra $\mathfrak{p} = \mathrm{Vect}(M) \ltimes C^\infty(M)$ has a simple geometric meaning: It is the Lie algebra of differential operators of the first order on M. Such an operator is always the sum $L_v + \rho$, where L_v is the operator of the Lie derivative along the field v on M, and ρ is regarded as the operator of the 0th order, namely the operator of multiplication by the function ρ.

Proposition 2.3 [GS1, MRW, Nov2]. *The equation of a barotropic fluid is a Hamiltonian equation on $\mathfrak{p}^*$ with respect to the linear Poisson–Lie structure and Hamiltonian function*

$$H(v, \rho) = -\int_M \left(\frac{1}{2} \rho v^2 + \Phi(\rho) \right) \mu,$$

where $\frac{d}{d\rho} \Phi(\rho) = h(\rho)$.

Remark 2.4. In contrast with the Euler dynamics (both of the rigid body and the ideal fluid), the total energy of a barotropic fluid is not a quadratic form, and it no longer has the meaning of a Riemannian metric on an appropriate group. However, one still has a variational problem on the cotangent space T^*P of the Lie group P, such that its extremals are the solutions of equations (2.1). The group-theoretical interpretation and all the Hamiltonian properties of the equations described earlier will be valid for the barotropic fluid (or gas dynamics) with merely cosmetic changes.

Note that for the one-dimensional manifold $M = \mathbb{R}$ or S^1 the equations of gas dynamics (2.1) for the algebra $\mathfrak{p} = \mathrm{Vect}(M) \ltimes C^\infty(M)$ are integrable (see Section 3.B). Note that this Lie algebra has three independent nontrivial 2-cocycles (one of them being the Gelfand–Fuchs cocycle of the Virasoro algebra).

Proof sketch of Proposition 2.3. One readily verifies the following

Proposition 2.5. *The dual to the space of vector fields* Vect(M) *on an n-dimensional manifold M is the space* $\Omega^1(M) \otimes_f \Omega^n(M)$, *where $\otimes_f$ means that the tensor product is taken over functions on M.*

In other words, elements of $\Omega^1(M) \otimes_f \Omega^n(M)$ are pairs $\beta \otimes \mu$, $\beta \in \Omega^1$, $\mu \in \Omega^n$, and we do not distinguish between the pairs $f\beta \otimes \mu$ or $\beta \otimes f\mu$ for all functions f.

The pairing between $v \in$ Vect(M) and $\bar\beta = \beta \otimes_f v \in \Omega^1(M) \otimes_f \Omega^n(M)$ is as follows:

$$\langle v, \, \beta \otimes \mu \rangle = \int_M (i_v \beta)\mu$$

(the vector field v is contracted with the 1-form β, and the obtained n-form $(i_v\beta)\mu$ is integrated over M). That this choice of dual space is natural is due to the (readily verified) fact that the coadjoint action of the Lie algebra Vect(M) is geometric:

(2.2) $$\text{Ad}^*_\varphi(\beta \otimes_f \mu) = \varphi_* \beta \otimes_f \varphi_* \mu;$$

i.e., it is given by a change of coordinates in both of the 1-form β and the n-form μ.

In case of the Lie algebra $\mathfrak{p}^* = $ Vect$(M) \ltimes C^\infty(M)$, elements of the corresponding dual space $\mathfrak{p}^*$ are pairs $(\bar\beta, \theta)$, where $\bar\beta \in \Omega^1(M) \otimes \Omega^n(M)$ and $\theta \in \Omega^n(M)$. We leave it to the reader to check that the coadjoint action of an element $(\varphi, a) \in$ Diff$(M) \ltimes C^\infty(M)$ is

$$\text{Ad}^*_{(\varphi,a)}(\bar\beta, \theta) = (\varphi_* \bar\beta + da \otimes \varphi_* \theta, \varphi_* \theta)$$

(see, e.g., [MRW]).

Once the coadjoint action is known, it is routine to find the variational derivative of the Hamiltonian function (see Definition 1.21) and the corresponding Euler equation, according to the general rule

$$\dot m = \text{ad}^*_{\delta H/\delta m} m.$$

$\square$

It turns out that the equations of barotropic fluid or gas dynamics have plenty of similarities with the incompressible case (e.g., the structure of conservation laws in even and odd dimensions is the same). This phenomenon is due to "incompressibility" of the barotropic fluid in coordinates moving with density.

Namely, let μ be the volume form on M induced by the Riemannian metric. Assign to the density function ρ the *density n-form* $\theta := \rho\mu \in \Omega^n(M)$.

Theorem 2.6 [KhC]**.** *The barotropic fluid equations (2.1) admit first integrals*

$$I(v) := \int_M u \wedge (du)^m \qquad and \qquad I_f(v) := \int_M f\left(\frac{(du)^m}{\theta}\right)\theta$$

according to the parity of $n = \dim(M)$ *(*$n = 2m + 1$ *and* $n = 2m$, *respectively), where the vector field* v *and the 1-form* u *are related by means of the metric, and* $f : \mathbb{R} \to \mathbb{R}$ *is an arbitrary function.*

The integrals above can be read off from (I.9.2) if one replaces the n-form μ by the density form $\theta = \rho\mu \in \Omega^n(M)$, with ρ being the density function. We shall show that these invariants are Casimir functions on the dual space to the Lie algebra $\mathfrak{p}$. Another (though trivial) conservation law of the same nature is given by the total mass of the fluid, that is, by the integral of the density form θ over the manifold M. The Hamiltonian function H is also a first integral of the equation, but it is not a Casimir function.

Remark 2.7. The equations of a barotropic fluid with a nearly constant density ρ approximate the Euler equation of an incompressible fluid [Eb]. One can think of the condition of incompressibility within the general framework of systems with constraints (see [Arn16]). A dynamical system confined to a submanifold can be regarded as a subsystem in an ambient manifold with a strong "returning force" directed towards the submanifold.

For instance, consider a point mass that is constrained to move in the unit circle in the plane without forces. It can be thought of as a point attached to the center by a rigid rod. The latter is the limiting case of a point attached to the center by an elastic spring, where the elasticity coefficient of the spring tends to infinity, and in equilibrium the spring has length 1. While a point on a rod is confined to a circle, a point on a spring oscillates out from this circle. In the limit, the position and velocity of the "elastic pendulum" tend to those of the "rigid pendulum," but the acceleration does not.

Similarly, for the group of all diffeomorphisms of a manifold, one can introduce a "returning force" directed towards the subgroup of all volume-preserving diffeomorphisms (see [Eb]). Then the velocity and its first partial derivatives of a barotropic (weakly compressible) fluid tend to those of an ideal fluid. In particular, the above conservation laws for a barotropic fluid become the conserved charges (I.9.2) for an ideal fluid as $\rho \to 1$. Indeed, their explicit form involves only the fluid velocity v and its first derivatives $\partial v/\partial x$ (or the corresponding 1-form u and its differential du, where u is related to v by means of the Riemannian metric; i.e., without any differentiation). The conservation laws do not contain time derivatives of the velocity (i.e., do not contain the acceleration), and hence the limiting procedure is harmless for them.

Proof of Theorem 2.6. A heuristic argument is based on the fact that the density ρ is transported by the flow and the fluid is incompressible with respect to the new volume form θ (depending on time and on the initial conditions). Thus, we can apply Theorem I.9.2, whose assumptions require no relation between the metric and the volume form.

More precisely, the trajectories of the barotropic fluid equations are tangent to the orbits of the coadjoint representation of the group $P = \text{Diff } M \ltimes C^\infty(M)$, and the statement follows from

Proposition 2.8. *The functional*

$$I(\bar{\beta}, \theta) = \int_M u \wedge (du)^m$$

in the case of an odd $n = 2m + 1$ and the functionals

$$I_f(\bar{\beta}, \theta) = \int f\left(\frac{(du)^m}{\theta}\right)\theta$$

in the case of an even $n = 2m$ (where the 1-form u is defined by $u := \bar{\beta}/\theta \in \Omega^1(M)$) are invariant under the coadjoint action of the group P on the dual space $\mathfrak{p}^*$.

Proof. Note that the ratio $u = \bar{\beta}/\theta$ has the geometric meaning of a differential 1-form (see (2.2)). Explicitly, one has the following action on this form:

$$\text{Ad}^*_{(\varphi,a)}\, u = \text{Ad}^*_{(\varphi,a)}\left(\frac{\bar{\beta}}{\theta}\right) = \frac{\varphi_*\bar{\beta} + da \otimes \varphi_*\theta}{\varphi_*\theta}$$

$$= \varphi_*\left(\frac{\bar{\beta}}{\theta}\right) + da = \varphi_*u + da;$$

i.e., the 1-form u is transported by the flow modulo da, the differential of a function. Hence, the P-action on the coset $[u] \in \Omega^1/d\Omega^0$ of 1-forms on M, as well as on the n-form $\theta \in \Omega^n$, is geometric: It is nothing but a change of variables. Now Proposition 2.7 (as well as Proposition I.9.3) follows from the coordinate-free definition of the functionals I and I_f. ☐

To complete the proof of the theorem, recall that the inertia operator $\tilde{A} : \mathfrak{p} \to \mathfrak{p}^*$ defined by the Riemannian metric on the manifold M is the map $(v, \rho) \mapsto (u \otimes \theta, \theta)$, where $\theta = \rho\mu$ is the density form on M, and the 1-form u is obtained from the velocity v by the metric "lifting indices." Theorem 2.6 follows. ☐

2.B. Other conservative fluid systems

We refer to the surveys [GS2, HMRW, MRW, Nov2, DKN, VlM] for extended treatments of the Hamiltonian formalism related to the variety of different types of fluids, and in particular for applications of the techniques of semidirect products and Hamiltonian reductions.

We mention just a few examples:

— A general inviscid compressible fluid is regarded as having two internal degrees of freedom: The pressure term is defined by both the mass density and the entropy (unlike the barotropic case with density only); see [Nov2].

The corresponding Euler equation is related to the semidirect product Lie algebra

$$\tilde{\mathfrak{p}} := \mathrm{Vect}(M) \ltimes [C_\epsilon^\infty(M) \oplus C_\rho^\infty(M)].$$

— Anisotropic liquids (say, superfluid ^{4}He) require the introduction of a vector field for the internal degrees of freedom [Nov2, KhL].

— Magnetohydrodynamics in a compressible perfectly conducting fluid is constructed as the semidirect product of the magnetic extension of the diffeomorphism group (considered in Section I.10) with the space of smooth functions on the manifold M; see [MRW].

— The motion of an ideal incompressible fluid with a free boundary does not have an explicit group structure: One cannot compose two flow transformations with different shapes of the boundary. The Hamiltonian formalism for this problem, as well as the Hamiltonian form for the equations of a liquid drop with surface tension, is presented in [LMMR].

— A rigid body in a fluid is described by the Kirchhoff equations in $\mathbb{R}^6$ (see Section I.10). However, the whole "body–fluid" system is already an infinite-dimensional system. The body floating in the fluid is described by its impulse and angular momentum, while the fluid can be regarded as an infinite-dimensional system of the above type (having one fixed boundary component and the other a "free" one). A fluid filling a cavity M in a body is another, similar, system. Its dynamics is associated to the semidirect product of the group $E(3)$ (the motion of the body) and $S\,\mathrm{Diff}(M)$ (the motion of the fluid filling the cavity). See [VlI] for the stability analysis corresponding to the systems of both types.

— The geometry of geodesics with respect to the H^1-metric on the group of volume-preserving diffeomorphisms and its relation to the averaged Euler equation is described in [HKMRS].

— Various equations related to two-dimensional hydrodynamics manifest some features of integrability. For instance, the Kadomtsev–Petviashvili equation $(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$ is an integrable infinite-dimensional Hamiltonian system related to shallow water.

— The equations of infinite conductivity (or those of the β-plane in meteorology: $\Delta\psi_t + \beta\psi_x + \{\psi, \Delta\psi\} = 0$) differ from the standard incompressible 2-D or 3-D hydrodynamics by a Coriolis-type term; see [Fey] and Section 2.C below.

— The equation $\{\psi^s + cy, \Delta\psi^s + \beta y\} = 0$ for steady waves in two dimensions, which is obtained from the β-plane equation by substituting $\psi(x, y, t) = \psi^s(x - ct, y)$, admits interesting solutions of steadily traveling dipole vortices [LaR] (see Section I.11.A for $\beta = 0$).

— Many dynamical systems on the sine-algebra, being the "quantum" version of the algebra of Hamiltonian fields on the two-torus (see Remark I.11.6), are described in [HOT].

— General Poisson brackets of hydrodynamic type [D-N, DKN] provide a
general Hamiltonian formalism for first-order quasilinear equations on
manifolds. The properties of these brackets impose very restrictive condi-
tions on the Riemannian structure of the underlying manifold.

One more advantage of the Hamiltonian approach is a simple geometric interpre-
tation of the so-called *Clebsch variables* in many physically interesting systems.
These variables appeared in a hydrodynamical setting as a set of an excessive
number of coordinates (with additional constraints between them) in which the
Euler equation acquires the canonical Hamiltonian form; see [Lam]. A general
framework for symplectic (or "Clebsch") variables from the Poisson point of view
can be found in [M-W] (see also [Zak, MRW]).

Definition 2.9 [M-W, MRW]**.** If P is a Poisson manifold, then *symplectic vari-
ables for P* is a map $J : M \to P$ of a symplectic manifold M into P that respects
the Poisson brackets (i.e., the pullback of the Poisson bracket of two functions f, g
on P is the Poisson bracket on M of their pullbacks $f \circ J$, $g \circ J$). Any canonical
symplectic coordinates on M are said to be *canonical coordinates* on the Poisson
manifold P.

A Hamiltonian function $H: P \to \mathbb{R}$ determines a Hamiltonian function on M by
$H_M := H \circ J$, and the integral curves of the "canonical" Hamiltonian system on M
with the Hamiltonian H_M cover those for the Poisson "noncanonical" Hamiltonian
system on P.

Example 2.10. The construction of the manifold M and the map J in the case of
the dual space $P = \mathfrak{g}^*$ to an arbitrary Lie algebra $\mathfrak{g}$ equipped with the Lie–Poisson
bracket is very explicit. The symplectic manifold becomes the cotangent bundle
$M = T^*G$ to the Lie group G, while the map J is the left shift L_g^* of any covector
$\xi \in T_g^*G$ at a point $g \in G$ to the cotangent space at the identity: $T_e^*G = \mathfrak{g}^*$. The
natural coordinates (p, q) in the cotangent bundle T^*G are canonical for $\mathfrak{g}^*$, since
the symplectic structure has the form $dp \wedge dq$.

A linear version of the variables on T^*G is the set of canonical coordinates on
$\bar{M} = \mathfrak{g}^* \oplus \mathfrak{g}$ with the map

$$\bar{J} : \mathfrak{g}^* \oplus \mathfrak{g} \to \mathfrak{g}^*$$

such that $(p, q) \mapsto \mathrm{ad}_q^* p$. We refer to [M-W, MRW, Zak] for a detailed description
and numerous applications of this construction of Clebsch variables to dynamical
systems and their conservation laws.

2.C. *Infinite conductivity equation*

The infinite conductivity equation possesses many properties inherent in ideal
hydrodynamics. Its relationship to the equation of an incompressible fluid is due
to the fact that at a high density, an electron gas is similar to a fluid. Indeed, the
repelling of particles in electron clusters makes the gas incompressible.

Definition 2.11 (see, e.g., [Fey]). The equation of (nonrelativistic) *infinite conductivity* in a domain of $\mathbb{R}^3$ is

$$(2.3) \qquad \dot{v} = -(v, \nabla)v - v \times \mathbf{B} - \nabla p,$$

where v denotes a divergence-free velocity field of the electron gas, $\mathbf{B}$ is a constant in time (but not in space) external divergence-free magnetic field, and the symbol $\times$ stands for the cross product in $\mathbb{R}^3$. One can define an analogue of this equation on an arbitrary Riemannian manifold M with volume form μ.

Proposition 2.12 [KhC]. *The infinite conductivity equation (2.3) is equivalent to the following Hamiltonian equation on the dual space $S \operatorname{Vect}(M)^* = \Omega^1(M)/d\Omega^0(M)$ to the Lie algebra of divergence-free vector fields $S \operatorname{Vect}(M)$:*

$$(2.4) \qquad \frac{\partial[u]}{\partial t} = -L_v[u + \alpha].$$

Here the 1-form u is related to the vector field v by means of the metric inertia operator, $[u] \in \Omega^1(M)/d\Omega^0(M)$ is the coset of the 1-form u, and α is a 1-form whose differential $d\alpha$ obeys the identity $d\alpha = -i_{\mathbf{B}}\mu$.

Proof. The proof follows just as in the ideal incompressible case considered in Chapter I (see equation (I.7.11)). The form α defined by $d\alpha = -i_{\mathbf{B}}\mu$ (up to the differential of a function) is precisely chosen to fit the term $v \times \mathbf{B}$ with the cross product in (2.3).

The infinite conductivity equation (2.3) is Hamiltonian, with the Hamiltonian function being (minus) the quadratic energy form shifted away from the origin of $S \operatorname{Vect}(M)^*$:

$$-H([u]) = -\frac{1}{2} \int_M (u + \alpha, u + \alpha)\mu.$$

The Euler equation corresponding to the latter function has the form

$$\frac{\partial[u + \alpha]}{\partial t} = -L_v[u + \alpha],$$

which is equivalent to (2.4). Indeed, the field $\mathbf{B}$ is constant in time, and hence

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\partial \alpha}{\partial t} = 0.$$

$\square$

Corollary 2.13. *The infinite conductivity equation (2.3) has either at least one or infinitely many first integrals, according to the parity of $n = \dim(M)$. The integrals are given by $I(v)$ and $I_f(v)$ in formula (I.9.2) with u replaced by $u + \alpha$ and where the 1-form α is as defined above.*

Remark 2.14. The equation of infinite conductivity (and its generalization to an n-dimensional manifold M) can be regarded as the Euler equation on the central

extension of the Lie algebra of divergence-free vector fields on M [Rog, Ze2]. The corresponding two-cocycle, extending the Lie algebra of divergence-free vector fields $S\,\mathrm{Vect}(M)$, is the Lichnerowicz 2-cocycle [Lich]: For any closed 2-form β on M,

$$c_\beta(v, w) = \int_M (i_w i_v \beta)\mu;$$

cf. Remark I.11.6 on the extension of the sine-algebra and the algebra of Hamiltonian vector fields on a two-dimensional torus.

§3. Kähler geometry and dynamical systems on the space of knots

Infinite-dimensional spaces of curves appear in the hydrodynamical setting as certain special "low-dimensional" coadjoint orbits of the diffeomorphism group of $\mathbb{R}^3$. This point of view connects many seemingly unrelated symplectic and Poisson varieties and dynamical systems on them.

3.A. Geometric structures on the set of embedded curves

Consider the *space $\mathcal{C}$ of smooth embedded nonparametrized oriented closed curves* (or the *space of knots*) in Euclidean three-dimensional space $\mathbb{R}^3$. It can be thought of as the set of all smooth maps $\gamma : S^1 \to \mathbb{R}^3$ of the circle into $\mathbb{R}^3$ such that γ is an immersion ($\gamma'(x) \neq 0$ for all $x \in S^1$), γ has no double points, and where any two maps with the same image are indistinguishable:

$$\mathcal{C} = \{\gamma : S^1 \to \mathbb{R}^3 \mid \gamma'(x) \neq 0 \,\forall x \in S^1, \ \gamma(x) = \gamma(y) \text{ iff } x = y\}/\gamma \sim (\gamma \circ \phi).$$

Here ϕ runs over all diffeomorphisms of the circle S^1.

Connected components of $\mathcal{C}$ are the classes of equivalent (oriented) knots. We will call two knots *equivalent* if there is an isotopy of the ambient space $\mathbb{R}^3$ sending one of the knots into the other. Locally constant functions on $\mathcal{C}$ are called the *knot invariants*.

The space of knots $\mathcal{C}$ can be equipped with a natural symplectic structure. Consider an embedded curve $\gamma = \gamma(S^1) \subset \mathbb{R}^3$. A tangent vector v to $\mathcal{C}$ at γ is an infinitesimal variation of the curve γ, that is, a normal vector field attached to $\gamma(S^1)$. In parametrized form the vector $v(x)$ is orthogonal to $\gamma'(x)$ in $\mathbb{R}^3$ for all $x \in S^1$.

Definition 3.1. The *(Marsden–Weinstein) symplectic structure* on the space of knots is the 2-form β on $\mathcal{C}$ whose value on the pair of elements $u, v \in T_\gamma \mathcal{C}$ is the oriented volume of the following collar along the curve γ. At every point $\gamma(x)$ the vectors $u(x)$ and $v(x)$ span a parallelogram, and the collar is the union of these parallelograms along $\gamma \subset \mathbb{R}^3$; see Fig. 74.

For a chosen parameter $x \in S^1$ one has

$$\beta(v, w) = \int_{S^1} \text{vol}\left(\gamma'(x),\ v(x),\ w(x)\right) dx,$$

where vol is the volume of the parallelepiped spanned by the three vectors. The integral clearly does not depend on the parametrization.
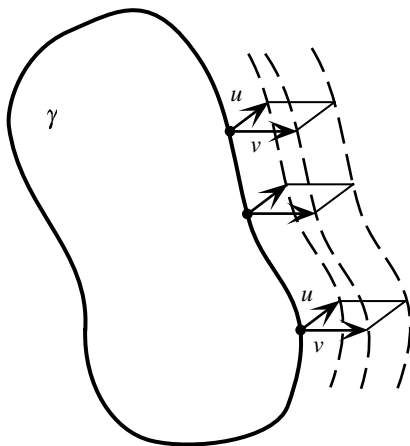
FIGURE 74.   The value of the symplectic structure on two variations of a knot is the volume of the collar spanned by the variations.

Note that we do not need the Euclidean structure but only the volume form in $\mathbb{R}^3$. The definition can be easily generalized to an arbitrary three-dimensional manifold with a volume form. Moreover, for manifolds of any dimension $n \geq 2$ the same definition gives the symplectic structure on the space of submanifolds of codimension 2 (i.e., of dimension $n - 2$).

The symplectic structure described above has a hydrodynamical meaning. It is based on the fact that every connected component of the space $\mathcal{C}$ (i.e., every isotopy class of knots) can be viewed as a special coadjoint orbit of the group of volume-preserving diffeomorphisms of $\mathbb{R}^3$.

Definition 3.2. Let γ be a knot in $\mathbb{R}^3$. Then it defines the *functional* ℓ_γ on divergence-free vector fields in the space: The value of ℓ_γ on a field v is the flux of the field v across any oriented surface in $\mathbb{R}^3$ bounded by the contour γ (such an embedded surface σ is called a Seifert surface).

Proposition 3.3. *The knot functional ℓ_γ is well-defined on divergence-free vector fields; i.e., its value does not depend on the choice of the surface σ such that $\partial\sigma = \gamma$.*

Proof. The difference between the fluxes of a field v through two surfaces with the same boundary γ is the flux of v across a closed surface. The latter vanishes by virtue of the divergence-free property of the field v. ☐

Remark 3.4. We now relate the functional $\ell_\gamma \in S\,\mathrm{Vect}(\mathbb{R}^3)^*$ to another description of the dual space as the quotient $S\,\mathrm{Vect}(\mathbb{R}^3)^* = \Omega^1(\mathbb{R}^3)/d\Omega^0(\mathbb{R}^3) = Z^2(\mathbb{R}^3)$ of all 1-forms on $\mathbb{R}^3$ modulo exact 1-forms, or as the space of all closed 2-forms. The exterior derivative d takes a coset of 1-forms (an element of $\Omega^1/d\Omega^0$) to a closed 2-form (an element of Z^2) without any loss of information, since $H^1(\mathbb{R}^3) = 0$; see Corollary I.7.9.

The curve γ is identified with a singular 2-form ω_γ in $\mathbb{R}^3$ supported on γ. It is a δ-type form whose integrals over any piece of a two-dimensional surface vanish, unless the piece intersects the curve. In the latter case, the integral equals the algebraic number of the intersection points, where the points are counted according to orientation determined by the orientation of the curve γ and the orientation of the piece at every point of intersection.

The 2-form ω_γ is closed, which corresponds to the closedness of the curve γ itself. Thus ω_γ belongs to $Z^2(\mathbb{R}^3)$ (more precisely, it is a so-called *De Rham current*, and it belongs to a certain closure of the space of smooth closed 2-forms; see [DeR]). To represent the closed (and hence, exact) 2-form ω_γ on $\mathbb{R}^3$ as an element of the quotient $\Omega^1/d\Omega^0$, we have to take d^{-1} of it. A 1-form $d^{-1}\omega_\gamma$ is not uniquely defined, and it can be thought of as the δ-type 1-form supported on a Seifert surface σ of the curve γ (Fig. 75). The coset of such a 1-form u_σ belongs to (a certain closure of) the space $\Omega^1/d\Omega^0$.
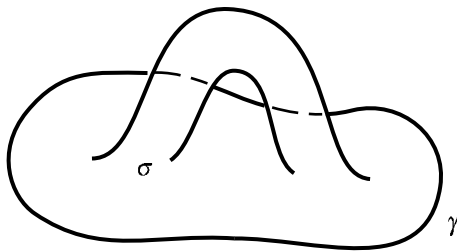
FIGURE 75. The 2-form ω_γ is supported on the curve γ. The 1-form $d^{-1}\omega_\gamma$ is supported on a Seifert surface σ.

Proposition 3.5. *The pairing of the 1-form u_σ with a divergence-free vector field v, according to the rules of Chapter I (see formula (I.7.7)), coincides with the flux of the field v across the surface σ.*

Proof. Let μ be a volume form in the space $\mathbb{R}^3$. Then the pairing of the (coset of the) 1-form u_σ and a divergence-free field v is

$$\langle [u_\sigma],\, v \rangle = \int_{\mathbb{R}^3} (i_v u_\sigma)\mu = \int_{\mathbb{R}^3} u_\sigma \wedge i_v \mu = \int_\sigma i_v \mu.$$

The last integral is a coordinate-free expression for the flux of v across σ.   □

We have identified knots with certain points in the dual space $S\,\mathrm{Vect}(\mathbb{R}^3)^*$. The coadjoint action of volume-preserving diffeomorphisms on knots is geometric, and hence all knots isotopic to a given one constitute a coadjoint orbit. The same consideration is applicable to links as well. Thus, the classification of knot invariants, though difficult enough by itself, becomes part of a much more complicated question on classification of all Casimir functions for the group of volume-preserving diffeomorphisms of the space $\mathbb{R}^3$ (or of the three-dimensional sphere S^3).

Proposition 3.6 [M-W]. *Identify the set of isotopic knots with a coadjoint orbit of the group of volume-preserving diffeomorphisms of $\mathbb{R}^3$. Then the Kirillov–Kostant symplectic structure on this set coincides with the Marsden–Weinstein symplectic structure.*

Proof. Assume that two fields v and w in $\mathbb{R}^3$ define two variations of a curve γ. Then the Kirillov–Kostant symplectic structure on the coadjoint orbit at the point ω_γ associates to these variations the number

$$\langle \omega_\gamma, \quad [v, w] \rangle := \langle d^{-1}\omega_\gamma, [v, w] \rangle = \langle u_\sigma, -\{v, w\} \rangle$$
$$= -\int_{\mathbb{R}^3} (i_{\{v,w\}} u_\sigma)\mu = -\int_{\mathbb{R}^3} u_\sigma \wedge (i_{\{v,w\}}\mu).$$

Here we have used the fact that the commutator in the Lie algebra of vector fields is equal to minus their Poisson bracket. Since $\{v, w\} = -\mathrm{curl}(v \times w)$, we have, according to the definition of $\mathrm{curl}(v \times w)$,

$$-i_{\{v,w\}}\mu = i_{\mathrm{curl}(v\times w)}\mu = d\alpha.$$

Here α is the 1-form related to the vector field $(v \times w)$ by means of the Riemannian metric: $\alpha(\xi) = (v \times w, \xi)$ for any vector field ξ. Then

$$\langle \omega_\gamma, [v, w] \rangle = \int_{\mathbb{R}^3} u_\sigma \wedge d\alpha = \int_{\mathbb{R}^3} du_\sigma \wedge \alpha = \int_{\mathbb{R}^3} \omega_\gamma \wedge \alpha.$$

The last integral, by definition of the 1-form α, is the circulation of the field $v \times w$ along γ, or, equivalently, the volume of the collar spanned by the variations v and w of the curve γ:

$$\int_{\mathbb{R}^3} \omega_\gamma \wedge \alpha = \int_{S^1} \mathrm{vol}\left(\gamma', v, w\right) dx = \beta(v, w).$$

This is the symplectic structure given in Definition 3.1.   □

Remark 3.7. Note that the coadjoint orbits corresponding to knots satisfy a kind of "quantization" condition. Associate to every knot a narrow current supported in a tubular neighborhood of the knot and whose flux across any transverse to the neighborhood is equal to 1. The flux of this current across a Seifert surface of any other knot is an integer.

If m is a point of such a "quantized" orbit, then the orbit of the point λm ($\lambda \in \mathbb{R}$) for a nonintegral λ does not correspond, in general, to a (nonparametrized) knot. It follows from the description of the coadjoint orbits as of the cosets $[\alpha]$ of 1-forms modulo differentials: The form $\lambda \cdot \alpha$ corresponds to the orbit λm. These knot-type orbits of the coadjoint representation depend on the form period $\oint \alpha$ as a parameter. The orbits of nonparametrized knots correspond to the 1-forms of period 1 in the description via cosets.

Consider the coadjoint orbit of one such link or knot. The considerations above imply that this "manifold" has the following peculiar property: The values of the "coordinates" of its points, equal to the linking numbers with other knots, are always integers, except for those knots that intersect the given one. In this sense the orbit is similar to a polyhedron whose faces are parallel to the coordinate subspaces and have integer-valued projections along these subspaces. The simplest example of a polyhedron of this type is a broken line on a chessboard consisting of parts of the square boundaries.

In general, one can think of these elements as a certain subset of the dual space, somewhat similar to the set of those points in a vector space with at least one coordinate being an integer. Presumably, replacing the integral coefficients of the knots forming the links by the rational ones, one obtains a set that is dense in some reasonable topology.

Remark 3.8. The space $\mathcal{C}$ can also be endowed with a natural almost complex structure: a continuous operator field $J_\gamma : T_\gamma \mathcal{C} \to T_\gamma \mathcal{C}$ such that $J_\gamma^2 = -1$ for all $\gamma \in \mathcal{C}$. This operator has a simple geometric meaning: Every variation field v along the curve γ is sent to another field Jv along γ whose vectors $Jv(x)$ at each point are obtained from $v(x)$ through a rotation by $\pi/2$ in the positive direction in the plane normal to $\gamma'(x)$ (see Fig. 76).

It turns out that the curvature tensor of this structure vanishes [PeS, Bry1]. In finite dimensions, this condition would be enough to introduce complex coordinates on the manifold (using the Newlander–Nirenberg theorem, [N-N]). However, the construction of complex coordinates does not carry over without restrictions to arbitrary infinite-dimensional manifolds. Here we deal with the infinite-dimensional manifold of all C^∞ curves, and one can show that it does not admit a complex structure [Lem, Wan]. According to V. Drinfeld and C. LeBrun, the situation is different in the category of analytic knots in an analytic manifold; see the discussion in [Bry1]. (We refer to [PeS] for a discussion of geometric quantization for vortex configurations.)

Note also that the moduli space of isometric maps of a circle into Euclidean space $\mathbb{R}^3$ (modulo orientation-preserving Euclidean motions) admits a complex structure [MiZ]. Another example of an infinite-dimensional complex manifold is given by a typical Virasoro coadjoint orbit; see [Ki3].

Remark 3.9. The above structures, as well as most of the dynamical systems we discuss below, can be defined on a bigger set $\bar{\mathcal{C}}$ of *immersed knots*, which has a nicer topology; see [Bry1]. The latter set is obtained by allowing the immersions
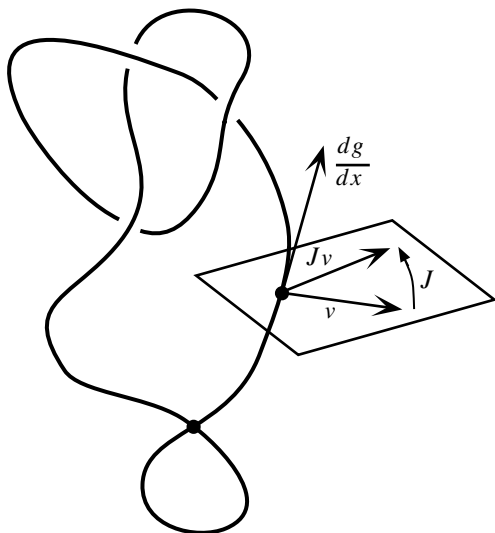
FIGURE 76. An immersed knot and the almost complex structure in the space of knots.

γ to have self-intersections in a finite number of points and of finite multiplicity (Fig. 76). The extension of the invariants of knots from the set $\mathcal{C}$ to the space of immersed knots $\bar{\mathcal{C}}$ is a cornerstone of the Vassiliev theory of knot invariants of finite order [VasV].

At first glance it seems that the space of singular knots with one self-intersection has (infinite) dimension that is one less than that of the symplectic space (the coadjoint orbit) of regular knots in the space of all circle immersions, and hence, it cannot carry a symplectic structure. This is, however, not the case. The corresponding coadjoint orbit has dimension two less than that of the regular knot: The singular knots with one double point (of a given topological type) form a two-parameter family of orbits (since the integral of the corresponding 1-form along each of the two loops is an invariant), while the regular knot orbits of a given topological type form a one-parameter family (the invariant being the integral along the whole knot).

3.B. Filament, nonlinear Schrödinger, and Heisenberg chain equations

To define a dynamical system on the (symplectic) space of nonparametrized (immersed) oriented knots $\bar{\mathcal{C}}$ (the closure of the set of embedded curves $\mathcal{C}$), we need a Hamiltonian function. A natural choice of the function H is the *length functional* on curves:

$$H(\gamma) := \text{length of } \gamma = \int_{S^1} \sqrt{(\gamma'(x),\ \gamma'(x))}\, dx.$$

Note that just as in ideal hydrodynamics, to define the motion we need to specify the Riemannian metric, in addition to the volume form on the manifold.

Definition 3.10. The evolution equation for the length Hamiltonian function H, with respect to the symplectic Marsden–Weinstein structure, is called the *filament equation*.

The time evolution $\gamma(x, t)$ of the curve $\gamma(x, 0)$, $x \in S^1$, according to the filament equation, is

$$(3.1) \qquad \frac{\partial \gamma}{\partial t} = k(x, t) \frac{\partial \gamma}{\partial x} \times \frac{\partial^2 \gamma}{\partial x^2},$$

where $k(x, t)$ is the curvature of the curve at the point x at time t.

Indeed, the variational derivative $\frac{\delta H}{\delta \gamma}$ is

$$-(\text{length of } \gamma)^{-1} \frac{d^2 \gamma}{dx^2}$$

for (a multiple of) arc-length parametrization x. Then, the corresponding Hamiltonian field can be found, say, by using the almost complex structure J_γ:

$$\text{sgrad } H = -(\text{length of } \gamma)^{-1} J_\gamma \left(\frac{\delta H}{\delta \gamma} \right) = (\text{length of } \gamma)^{-2} J_\gamma \left(\frac{d^2 \gamma}{dx^2} \right)$$

$$(3.1') \qquad = (\text{length of } \gamma)^{-2} \left(\frac{d\gamma}{dx} \times \frac{d^2 \gamma}{dx^2} \right).$$

The return from the arc-parametrization to an arbitrary one results in the curvature factor $k(x, t)$ in equation (3.1).

Remark 3.11. Hasimoto noticed in [Has] that the filament equation (3.1) is equivalent to the *nonlinear Schrödinger equation*

$$(3.2) \qquad -i \frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} |\psi|^2 \psi$$

for a complex-valued wave function $\psi : S^1 \to \mathbb{C}$. This equation is known to be a completely integrable infinite-dimensional system and to possess soliton solutions (see, e.g., [DKN]).

The transformation reducing one of the equations to the other is called the *Hasimoto transformation*:

$$\psi(x, t) = k(x, t) \exp \left(i \cdot \int_0^x \tau(u, t) \, du \right),$$

where $\tau(u, t)$ is the torsion of the curve γ at the point u and time t.

The paper [LaP] shows that the Hasimoto transformation respects the Hamiltonian property of the equations: It sends the Marsden–Weinstein structure on curves to a certain (nonconstant) Poisson structure on wave functions.

Remark 3.12. Another equivalent form of the filament equation is the *equation of gas dynamics* we dealt with in Section 2. Rewriting equation (3.1) in the Frenet frame of γ, one obtains the evolution equations on the curvature $k(x, t)$ and the torsion $\tau(x, t)$, which in the coordinates $\rho := k^2$ ("energy density" of the curve) and τ are

$$
\begin{cases}
\partial_t \rho + \partial_x (\rho \tau) = 0, \\
\partial_t \tau + \tau \partial_x \tau = \partial_x \left(\tfrac{1}{4} \rho + \tfrac{1}{2} \rho^{-1/2} \partial_x^2 \rho^{1/2} \right),
\end{cases}
$$

where $\partial_x := \partial/\partial x$ and $\partial_t := \partial/\partial t$; see [Tur]. Thus ρ and τ become the velocity and density fields for a one-dimensional fluid.

Remark 3.13. The *Heisenberg magnetic chain* provides one more version of the filament equation. This equation describes the dynamics of the vector function $L(x) \in \mathbb{R}^3$, $x \in S^1$:

$$
(3.3) \qquad \frac{\partial L}{\partial t} = L \times \frac{\partial^2 L}{\partial x^2}.
$$

One immediately obtains this equation from (3.1) by using the arc-parametrization x along the curve γ. Indeed, the filament equation (3.1–3.1′) assumes the form

$$
\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial x} \times \frac{\partial^2 \gamma}{\partial x^2},
$$

equivalent to equation (3.3) for the corresponding tangent vector $L := \partial \gamma / \partial x$.

The vector $L(x) \in \mathbb{R}^3$ can also be regarded as an element of the three-dimensional Lie algebra $\mathfrak{so}(3)$. From this point of view equation (3.3) is a particular case of the *Landau–Lifschitz equation*, which can be associated to an arbitrary finite-dimensional Lie group, or rather to the corresponding gauge group.

Remark 3.14. The filament equation can be regarded as an "approximation" of the Euler–Helmholtz equation for the vorticity concentrated on a curve if one considers the contribution of the local terms only; cf. Section I.11.C. The integrable dynamics in this case is a consequence of the approximation. The inclusion of the next (nonlocal) term into the picture makes the dynamics much more complicated; see [KlM].

3.C. Loop groups and the general Landau–Lifschitz equation

Let G be a finite-dimensional matrix group with a nondegenerate Killing form $\langle A, B \rangle = \mathrm{tr}(AB)$ for $A, B \in G$; i.e., a *reductive group* (one can think of $SO(3)$ in the example above, or the group of all nondegenerate matrices $GL(n)$), and let $\mathfrak{g}$ be the corresponding Lie algebra.

Definition 3.15. The *loop group* $\tilde{G}$ (or the *gauge group*) is the group of G-valued functions on the circle $\tilde{G} = C^\infty(S^1, G)$ with pointwise multiplication. The corresponding *loop Lie algebra* $\tilde{\mathfrak{g}}$ is the Lie algebra of $\mathfrak{g}$-valued functions on the circle with pointwise commutator.

Definition 3.16. The *Landau–Lifschitz equation* is the evolution equation

$$\partial_t L = L \times \partial_x^2 L$$

for a vector-valued function $x \mapsto L(x) \in \mathbb{R}^3$ on the circle $x \in S^1$ and $\partial_x^2 := \frac{\partial^2}{\partial x^2}$.

More generally, the *Landau–Lifschitz equation* associated to a Lie algebra $\mathfrak{g}$ is the following evolution equation:

$$(3.4) \qquad \partial_t m = [m, \ \partial_x^2 m],$$

where m is a $\mathfrak{g}$-valued function on S^1.

According to the latter definition, the classical Landau–Lifschitz equation (3.3) is associated to the Lie group $\mathfrak{so}(3)$ upon the identification of the vectors in $\mathbb{R}^3$ with angular velocities, the elements in $\mathfrak{so}(3)$:

$$L = (v_1, v_2, v_3) \mapsto \begin{pmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix}.$$

Theorem 3.17. *The Landau–Lifschitz equation associated to a Lie algebra $\mathfrak{g}$ is the Euler equation corresponding to the loop group $\tilde{G}$ with the quadratic Hamiltonian function*

$$H(m) = \frac{1}{2} \int_{S^1} \operatorname{tr} (\partial_x m)^2 \ dx$$

on the dual space $\tilde{\mathfrak{g}}^*$, where $\partial_x m$ is the $\mathfrak{g}$-valued function, and tr stands for the trace in the matrix algebra $\mathfrak{g}$.

Proof. The inverse inertia operator $A^{-1} : \tilde{\mathfrak{g}}^* \to \tilde{\mathfrak{g}}$ corresponding to this Hamiltonian sends a ($\mathfrak{g}^*$-valued) function m to the ($\mathfrak{g}$-valued) function $-\partial_x^2 m$. Then the Euler equation assumes the form

$$\partial_t m = \operatorname{ad}^*_{A^{-1}m} m = -[\partial_x^2 m, \ m],$$

equivalent to (3.4). $\qquad \qquad \square$

Corollary 3.18 (see, e.g., [A-L]). *The classical Landau–Lifschitz equation (3.3) is the Hamiltonian equation on the dual space $\tilde{\mathfrak{so}}(3)^*$ with the Hamiltonian function*

$$H(L) = - \int_{S^1} (\partial_x v_1)^2 + (\partial_x v_2)^2 + (\partial_x v_3)^2 \ dx$$

(here $L(x) = (v_1(x), v_2(x), v_3(x)) \in \mathbb{R}^3 = \mathfrak{so}(3)^*$).

The papers [A-L, Luk6] contain the calculations of the sectional curvatures of the loop group $\widetilde{SO}(3)$ with respect to the right-invariant Riemannian metric induced by the Hamiltonian function $H(L)$.

§4. Sobolev's equation

Studying fluid oscillations in a fast rotating tank, and starting with the corresponding approximating equation

$$(4.1) \qquad \frac{\partial v}{\partial t} - k(v \times e_z) + \operatorname{grad} p = F, \quad \operatorname{div} v = 0$$

(with unknowns v and p), S.L. Sobolev obtained an equation of unusual type, now named after him.

Definition 4.1. The *Sobolev equation* is the equation

$$(4.2) \qquad \frac{\partial^2 \Delta u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

for the unknown function u.

Remark 4.2. Equation (4.1) is the the linearization of the Navier–Stokes equation in a rotating domain. Typical examples are atmospheres of planets and fuel tanks of rotating projectiles. Poincaré [Poi2] reduced the linear system (4.1) to one equation (4.2). The latter equation was named after Sobolev, who rediscovered it in the forties and studied the corresponding boundary problems.

Sobolev's work was declassified and published in [Sob2]. This paper was in fact written in Kazan, perhaps in 1942. Sobolev's neighbor was Pontryagin, and they discussed many relevant problems in functional analysis. In particular, they considered the "pseudo-Hilbert" spaces with one (studied by Sobolev) or a finite number (studied by Pontryagin) of negative squares in the metric. These spaces are now called Pontryagin spaces. Very few people know that the theory of these spaces originated in the classified hydrodynamical work of Sobolev.

The work of Poincaré and of Sobolev was continued by Babin, Mahalov, and Nicolaenko, who extended the equation to the case of fast rotation and shallow domains, and considered nonlinear dynamics of the Navier–Stokes equation. Many features of Sobolev's study of the linear problem, such as the small denominators and the Diophantine incommensurability conditions on the domains' geometrical parameters, reappear in [BMN]. It is shown in [BMN] that solutions of the 3-D Euler and Navier–Stokes equations of uniformly rotating fluids can be decomposed into the sum of the following terms: a solution of the 2-D Euler (or Navier–Stokes) system with vertically averaged initial data, a vector field explicitly expressed in terms of the phases, and a small remainder.

Remark 4.3. To derive the Sobolev equation from the system (4.1) with $F = 0$ (see [GaS] for details) we take the curl of both sides of the first equation. Since

$\text{curl}(a \times b) = -\{a, b\}$, this gives

(4.3)
$$\frac{\partial \omega}{\partial t} + 2k \frac{\partial u}{\partial z} = 0, \qquad \text{where} \quad \text{curl } u = \omega.$$

Take the curl once more,

$$-\frac{\partial}{\partial t} \Delta u + 2k \frac{\partial \omega}{\partial z} = 0,$$

and differentiate it in t to get

$$-\frac{\partial^2}{\partial t^2} \Delta u + 2k \frac{\partial}{\partial z} \frac{\partial \omega}{\partial t} = 0.$$

Finally, substitute $\partial \omega / \partial t = -2k \partial u / \partial z$ from equation (4.3) to obtain the Sobolev equation

$$-\frac{\partial^2 \Delta u}{\partial t^2} - 4k^2 \frac{\partial^2 u}{\partial z^2} = 0.$$

A study of the spectral problems for the linear Sobolev equation showed a strong dependence of the eigen oscillations on the tank shape. Namely, after some transformations, Sobolev found it necessary to investigate the two-dimensional spectral problem

$$\frac{\partial^2 u}{\partial x^2} - \lambda \frac{\partial^2 u}{\partial y^2} = 0, \quad u|_\Gamma = g$$

(as well as the corresponding homogeneous problem $u|_\Gamma = 0$) in a plane domain bounded by a curve Γ.

For a given value of the spectral parameter λ, the equation above is the Dirichlet problem for the one-dimensional wave equation. Solving it by the method of characteristics, one immediately encounters the strong dependence of the results on the domain shape.

As we shall see below, on the boundary of the domain there appears a dynamical system. The ergodic properties of this system have a strong impact on the oscillation character.

Consider the case of a convex domain. Two families of characteristic lines cover the domain. Each of these two families defines the diffeomorphism of the boundary Γ into itself that is the involution exchanging the points of intersection of each characteristic with the boundary. The above-mentioned dynamical system on the boundary curve is the diffeomorphism of the curve Γ that is the composition of two involutions corresponding to the two families of characteristics.

In terms of this diffeomorphism $T : \Gamma \to \Gamma$, the solution of the above Dirichlet problem (for a fixed λ) is constructed as follows. First, by a linear change of variables, we transform the characteristics into the straight lines $x = \text{const}$ and $y = \text{const}$. Our problem assumes the form

$$\frac{\partial^2 u}{\partial x \, \partial y} = 0, \quad u|_\Gamma = g.$$

The solution u is the sum of two functions $f(x) + h(y)$, one of which depends only on x and the other only on y. To look for these functions, we fix some boundary point A and choose the value of one of the functions at this point (e.g., $f(A)$) arbitrarily. Then the value of the second function is determined by the boundary condition (i.e., $h(A) = g(A) - f(A)$).

Let B be the intersection of the characteristic line of the first family ($x = $ const) passing through A and the boundary Γ. At the point B we already know the value of the first function (it is the same as at the point A; i.e., $f(B) = f(A)$). Then the value of the second function h at B is determined by their sum $g(B)$ (namely, $h(B) = g(B) - f(B) = g(B) - f(A)$). Further, the characteristic of the second family ($y = $ const) passing through B intersects the boundary Γ at the point $A' = TA$ (Fig. 77). We already know the value of the second function along this line (which is the same as that at B: $h(A') = h(B)$). Given the sum $g(A')$, we find the value of the first function at A' (here $f(A') = g(A') - h(A') = g(A') - h(B) = g(A') - g(B) + f(A)$) and so on.
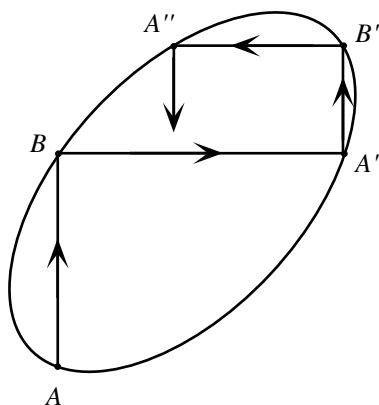
FIGURE 77.   Constructing the solution of the Dirichlet problem for the wave equation from two families of characteristics.

The infinite process of constructing the solution is described by a piecewise-linear trajectory. This trajectory is constituted by the intermittent segments of the characteristics joining the points $A^{(n)} = T^n A$. The solution is the sum of the initial value and the alternating sum of the boundary values at the vertices of the piecewise-linear trajectory.

The properties of the dynamical system $T : \Gamma \to \Gamma$ have the following impact on the solutions of our Dirichlet problem. Suppose that T has a periodic trajectory, $T^n A = A$. Then the alternating sum of values of the boundary function g at the vertices of the corresponding piecewise-linear trajectory $ABA'B' \cdots A$ must be equal to zero. Hence, each periodic trajectory of the map T corresponds to a solvability condition for the Dirichlet problem (and therefore, to a certain nontrivial

"distributional solution" of the corresponding homogeneous equation, "supported near" this periodic trajectory).

There are more subtle properties of the dynamics of T that also affect the solvability of the Dirichlet problem (see details, e.g., in [Arn2, FoP]).

Consider first an elliptic domain. In this case, the diffeomorphism T becomes a rotation after an appropriate choice of the angle coordinate on the boundary. Indeed, an ellipse can be turned into a circle by an affine transformation of the plane. The characteristics of both families will turn into two families of parallel lines forming an angle α with each other. The map T will become the circle rotation by the angle 2α (by virtue of the "inscribed angle" theorem).

Depending on whether the angle α is commensurable with 2π or not, the orbits of the rotation T either consist each of a finite number of points (repeating periodically with the same period for all initial points) or are everywhere dense on the circle.

In the first ("resonance") case, the solution of the nonhomogeneous equation exists if and only if the function g satisfies an infinite number of independent conditions. The corresponding homogeneous problem has an infinite-dimensional space of solutions.

When the angle α is not commensurable with 2π, any T-orbit is everywhere dense (it is the second, "ergodic," case). Here the situation is more complicated. Formally, one can find the solution as a Fourier series. However, its convergence relies on the arithmetic Diophantine properties of the irrational number $\alpha/2\pi$ (as well as in what functional space the problem is considered). For almost all (in the sense of Lebesgue measure) irrational numbers $\alpha/2\pi$, the corresponding homogeneous problem has the unique solution $u = 0$. The nonhomogeneous problem has, in this case, a (smooth) solution for every sufficiently smooth right-hand side g (the necessary smoothness of g increases as the required smoothness of the solution increases; for an analytic solution the analyticity of the right-hand side is sufficient).

The case of an ellipse, discussed above, is not generic, since the dynamics of the corresponding diffeomorphism T is integrable. (According to Yurkin [Yur] a domain bounded by ellipses is the only type of cavity in a rotating symmetric top for which the study of oscillations described by the Sobolev equation can be reduced to a finite-dimensional problem.) For a typical boundary curve the diffeomorphism T cannot, in general, be reduced to a rotation, no matter what angle coordinate on the curve is chosen.

In the space of all diffeomorphisms (and hence, in the space of curves Γ), the *structurally stable* diffeomorphisms form an open and everywhere dense set. Such diffeomorphisms are of "resonance type" with a finite number of periodic orbits (all of which have the same period) and alternating attractors and repulsers.

People working in the axiomatic theory of dynamical systems usually assume that "generic" events are those occurring on an everywhere dense open set in the space of systems. From this viewpoint "generic" circle diffeomorphisms are the structurally stable ones.

However, from the physics point of view, these structurally stable systems are not the most widespread. Consider, for instance, a family of circle diffeomorphisms $x \rightarrow x + a + b \sin x \pmod{2\pi}$, where a and b are parameters. For most of the points (a, b) in the rectangle $0 \leq a \leq 2\pi$, $0 \leq b \leq \beta$ of a sufficiently small height β, the diffeomorphism does not have periodic points at all, and one can make it into a rotation by choosing an appropriate coordinate on the circle. (This will be the rotation by an angle incommensurable with 2π.) Every orbit of such a diffeomorphism is everywhere dense on the circle. For almost all values of the rotation angle, the solvability question for the Dirichlet problem, corresponding to such a diffeomorphism T, turns out to be the same as that for the integrable case of an elliptic boundary.

For instance, for the near-elliptic domains the "ergodic" situations are encountered in an overwhelming majority of cases, while the "resonance" ones are rare (but form an open and everywhere dense set) in the space of ellipse deformations; see [Arn2].

We return now to the initial spectral problem with the parameter λ. For a typical boundary Γ, the two types of behavior of the dynamical system $T = T(\lambda)$ on the curve Γ alternate as λ changes. If Γ is a typical small perturbation of an ellipse, then the resonance values of the parameter λ (for which nontrivial eigenfunctions arise) form an infinite everywhere dense set (of small measure) on the axis λ. The ergodic values of λ (i.e., the values λ for which $T(\lambda)$ reduces to the circle rotation by a smooth coordinate change) form a Cantor-type set of almost full measure (for small perturbations of an ellipse).

As we can see, all topological subtleties of the nonlinear theory of dynamical systems (in particular, of their perturbation theory) appear in hydrodynamics in studying the spectrum of the *linear* problem of small oscillations of a fluid.

After S.L. Sobolev, the spectral problem was studied by R.A. Alexandryan and his school (see [Ale]). We mention the series of papers by S.G. Ovsepjan [Ovs], in which the case of a nonconvex boundary was treated.

In the nonconvex case a new difficulty arises: A characteristic line intersects the boundary in more than just two points, so that the "dynamics" T turns out to be multivalued (or branching). The ergodic properties of this branching multivalued dynamics form an interesting, but an insufficiently explored, part of the theory of dynamical systems.

Consider, for example, a circle diffeomorphism that becomes a multivalued algebraic correspondence of an algebraic curve into itself when the diffeomorphism is extended into the complex domain. This means that the graph of the diffeomorphism is one of the components of a real algebraic curve on the Cartesian square of another algebraic curve. One believes that the number of attractors of such a diffeomorphism is bounded by a constant, depending only on the discrete invariants of the correspondence (the genera of the curves and the degree of the correspondence). However, it has been proved only for the correspondences univalued in one of the directions (say, for polynomial or rational maps of the Riemann sphere into itself); see [Jak, Herm].

Remark 4.4. The Dirichlet problem for the one-dimensional wave equation is encountered in many problems of different origins. For instance, J.-P. Dufour [Duf] treated in detail its local analogue for algebraic curves with singularities (say, $x^2 = y^3$). This problem arises in the study of symmetry loss (for example, for the classification of Morse functions in a neighborhood of the fixed point of the line involution $x \to -x$, or for the classification of pairs of line involutions in a neighborhood of the common fixed point); see the survey of S. Voronin [Vor].

An analogous method was used in [Arn1] in the study of the representations of functions on trees by sums of functions of the coordinates, which is related to the 13th Hilbert problem. It is interesting that the main trick in all these problems is the composition of the alternating sums of values of a known function along a piecewise characteristic, and it is exactly the same as the one used in hydrodynamics in the study of spectral problems for the Sobolev equation.

§5. Elliptic coordinates from the hydrodynamical viewpoint

Imagine an electrically charged metallic ellipsoid. A theorem going back to Newton [NewI] and Ivory [Ivo] states that the potential (of the electrostatic field) induced by the charges is constant inside the ellipsoid, while outside of it the equipotential surfaces are the ellipsoids confocal to the initial one. As we shall see below (following [Arn12, ShV]), this fact, as well as its higher-dimensional generalizations, has a genuine hydrodynamical flavor: The electromagnetic fields of this type are generated by incompressible flows of electric charges along quadrics.

5.A. Charges on quadrics in three dimensions

We start with a quadric surface (say, ellipsoid) Q in three-dimensional space and include it first in the family of confocal quadrics.

Definition 5.1. For a quadric Q defined by the equation

$$\frac{x^2}{a_1} + \frac{y^2}{a_2} + \frac{z^2}{a_3} = 1,$$

the *confocal family of quadrics* $Q_{\mathrm{cnf}}(\lambda)$ is the following family of surfaces:

$$Q_{\mathrm{cnf}}(\lambda) = \left\{ \frac{x^2}{a_1 + \lambda} + \frac{y^2}{a_2 + \lambda} + \frac{z^2}{a_3 + \lambda} = 1 \right\}.$$

The quadrics of the family change signature at $\lambda = -a_1, -a_2,$ or $-a_3$. For instance, for a hyperboloid of one sheet with $a_1 > a_2 > 0 > a_3$ the family consists of the hyperboloids of two sheets for $-a_1 < \lambda < -a_2$, of the hyperboloids of one sheet for $-a_2 < \lambda < a_3$, and of the ellipsoids for $a_3 < \lambda$ (Fig. 78).

We will also use another family of quadrics containing our initial surface Q: quadrics *homothetic* to Q with center at the origin. First let Q be an ellipsoid.

Definition 5.2. A *homeoidal density* on the surface of an ellipsoid Q is the density of a layer between Q and an infinitely nearby ellipsoid homothetic to Q.

Now we can make mathematical sense of the "free distribution of electric charges" on the surface of an ellipsoid:

Theorem 5.3 (Ivory Theorem, see [Ivo, Arn12]**).** *A finite mass distributed on the surface of an ellipsoid with homeoidal density does not attract any internal point; it attracts every external point the same way as if the mass were distributed with homeoidal density on the surface of any smaller confocal ellipsoid.*

The attraction of the charges is defined by the Coulomb (or Newton) law: In $\mathbb{R}^n$ the force is proportional to r^{1-n} (as prescribed by the fundamental solution of the Laplace equation).

In the counterparts of Ivory's theorem for hyperboloids, one replaces the homeoidal densities on ellipsoids by harmonic forms of different degrees, and the Coulomb potential by the generalized potentials related to the Biot–Savart law.

In the simplest nontrivial case of a hyperboloid H of one sheet in three-dimensional Euclidean space, the result is as follows. Consider the intersection curves of the hyperboloid with other quadrics of the confocal family $H_{\text{cnf}}(\lambda)$. We will be referring to the intersections with confocal ellipsoids (respectively, confocal hyperboloids of two sheets) as *parallels* (respectively, *meridians*) of H. (Notice that the parallels and meridians are orthogonal to one another at every point of the hyperboloid; Fig. 78. This is the theorem on the existence of an orthogonal eigenbasis for a symmetric matrix, applied to the Legendre dual family of quadrics; see, e.g., [A-G].)
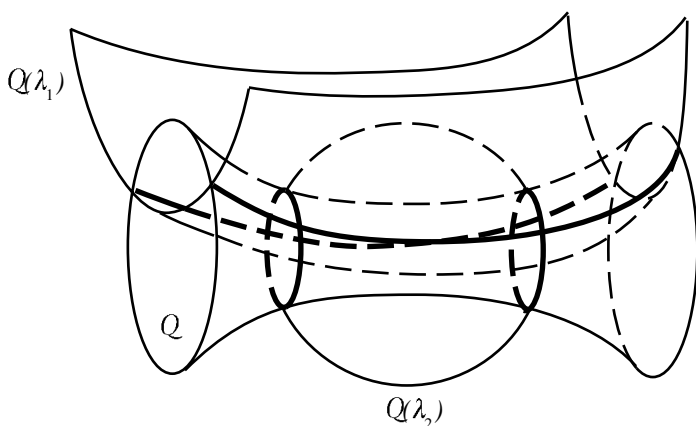
FIGURE 78. Quadrics of the confocal family intersect a hyperboloid along the orthogonal system of curves.

The hyperboloid H divides the space $\mathbb{R}^3$ into two parts, "internal" $I(H)$ and "external" $E(H)$, the latter being non-simply connected. The region inside the hyperboloidal tube is also smoothly fibered by meridians (orthogonal to the ellipsoids in the confocal family), while the annular region outside the hyperboloid is smoothly fibered by parallels (orthogonal to the hyperboloids of two sheets).

Theorem 5.4 [Arn12]. *There exists a unique (modulo a constant factor) surface current flowing along the meridians of the hyperboloid that produces a magnetic field that vanishes in the inner domain and is directed along parallels in the exterior domain of the hyperboloid. Similarly, there exists a unique (modulo a constant factor) surface current flowing along the parallels of the hyperboloid that produces a magnetic field that vanishes in the exterior domain and is directed along meridians in the inner domain of the hyperboloid.*

The magnetic field in the inner domain for the hyperboloid, but outside of a charged ellipsoid from the same confocal family, coincides modulo sign with the electrostatic field of the ellipsoid. Furthermore, let us look at the electrostatic field produced by two charges of opposite signs, "equal in magnitude," and distributed with homeoidal density on the surfaces of a conducting hyperboloid of two sheets. This field between the surfaces coincides (modulo sign) with the magnetic field in the exterior domain of a confocal hyperboloid of one sheet. The explicit formulas are given below.

Remark 5.5. The vector fields given by Theorem 5.4 are exact stationary solutions of the corresponding Euler equations of an incompressible fluid flowing, respectively, inside or outside of the hyperboloid in $\mathbb{R}^3$. The flow is potential in the inner domain of a triaxial hyperboloid, and it is vorticity-free in the exterior domain.

5.B. Charges on higher-dimensional quadrics

Let Q be a nondegenerate quadric centered at the origin of Euclidean n-dimensional space. Include it in the family of *confocal quadrics*

$$Q_{\mathrm{cnf}}(\lambda) := \left\{ \sum_{i=1}^{n} \frac{x_i^2}{a_i + \lambda} = 1 \right\},$$

as the hypersurface corresponding to $\lambda = 0$. Let us order the axes as follows: $a_n < \cdots < a_1$.

Definition 5.6. The *elliptic coordinates* of a point $x \in \mathbb{R}^n$ is the set of n values of λ (in increasing order) for which a quadric of the family $Q_{\mathrm{cnf}}(\lambda)$ passes through x. Note that it is an orthogonal coordinate system, since the confocal quadrics meet at right angles.

The results formulated above for the three-dimensional case have been extended by B. Shapiro and A. Vainshtein [ShV] to hyperboloids in Euclidean spaces of any number of dimensions. A nonsingular hyperboloid H in $\mathbb{R}^n$, diffeomorphic to $S^l \times \mathbb{R}^k$, divides the space into the exterior region $E(H)$ (diffeomorphic to the product of S^l with a half-space) and the interior $I(H)$.

Let ω be a differential form with distribution coefficients (see [DeR]). The form is said to be *harmonic off a hypersurface* Γ if it is continuous off this hypersurface, coclosed (i.e., $\delta\omega = 0$, where δ is the operator conjugate to the external derivative d; see Section V.3.B), and if its exterior derivative is a form (with distribution coefficients) supported on Γ.

Theorem 5.7 [ShV]. *Given a hyperboloid H there exists a unique (modulo a constant factor) l-form harmonic off H, decomposable in elliptic coordinates, and vanishing in the interior region $I(H)$, and there exists a unique (modulo a constant factor) k-form harmonic off H, decomposable in elliptic coordinates, and vanishing in the exterior $E(H)$.*

These forms are induced by certain homeoidal densities on the *focal quadrics*, which are the limiting quadrics of the confocal family, when the shortest axis of the hyperboloids or ellipsoids shrinks to zero. We refer to [ShV] for explicit formulas and proofs.

For the hyperboloids of indices $(1, n-1)$ and $(n-1, 1)$, one can give the following magnetohydrodynamical meaning to those densities.

Let H^1 be a nondegenerate quadric with $a_n < \cdots < a_2 < 0 < a_1$ (and a quadric H^{n-1}, with $a_n < 0 < a_{n-1} < \cdots < a_1$, respectively). Similar to the above, the exterior region $E(H^{n-1})$ for the hyperboloid H^{n-1} is fibered by *parallels* (diffeomorphic to a circle) by fixing the values of all $n-1$ elliptic coordinates *positive* in $E(H^{n-1})$. The inner domain $I(H^1)$ for the quadric H^1 is fibered by *meridians* (diffeomorphic to a line) by fixing the values of all $n-1$ elliptic coordinates *negative* in $I(H^1)$.

Theorem 5.7′ [ShV]. *There exists a unique (modulo a constant factor) potential flow of an incompressible fluid in the inner domain $I(H^1)$ whose trajectories coincide with the meridians. Similarly, there exists a unique (modulo a constant factor) flow of an incompressible fluid in the exterior region $E(H^{n-1})$ whose vorticity vanishes and the trajectories of which are the parallels.*

By construction, both of the flows are directed along the remaining elliptic coordinate. Say, in the 3-dimensional case, one has the following explicit formulas for the corresponding vector fields v_1 and v_2 in the regions $I(H^1)$ and $E(H^2)$, respectively, in the elliptic coordinates $\lambda_1 > \lambda_2 > \lambda_3$ (see [ShV]):

$$v_1 = \frac{\lambda_2 - \lambda_3}{\Phi(\lambda_2)\Phi(\lambda_3)} \frac{\partial}{\partial \lambda_1} \qquad \text{and} \qquad v_2 = \frac{\lambda_1 - \lambda_2}{\Phi(\lambda_1)\Phi(\lambda_2)} \frac{\partial}{\partial \lambda_3},$$

where

$$\Phi(\lambda_i) = \sqrt{(\lambda_i + a_1)(\lambda_i + a_2)(\lambda_i + a_3)}.$$

Noncomputational proofs of these geometric theorems are unknown, even in the three-dimensional case.

Question 5.8. The presence of distinguished forms that are harmonic off hyperboloids suggests that one might try to find filtrations, analogous to those arising in the theory of mixed Hodge structures, in spaces of differential forms on noncompact real algebraic and semialgebraic varieties.

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Index